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On an Inverse Problem of Identifying an Unknown Boundary for the Biharmonic Equation from Cauchy Data

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ABSTRACT

This paper deals with an inverse problem for the biharmonic equation to find an unknown boundary in the plane by using additional information assumed on the remaining known part of the boundary. As a by-product, we can uniquely determine the solution everywhere in its domain of definition by supposing that the available data have Fourier expansions. The question of the existence and uniqueness of this inverse problem will be investigated, and we will conclude with some analytical examples to ensure the validity of this study.

Keywords: Biharmonic Equation, Elastic Problem, Inverse Problem, Cauchy Problem.

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INTRODUCTION

In mechanics, physics, and many engineering applications, the biharmonic equation is used as a governing equation to describe: the deformation of thin plates, the motion of fluids, free boundary problems, non-linear elasticity, and problems related to blending surfaces. Several implementations of the 2D biharmonic problem have been consecrated in simply and multiply-connected regions (see, e.g., [3, 5, 1, 8, 9, 21, 13, 19, 22, 4, 10, 12]).

Boundary conditions are essential and extremely important constraints for solving a boundary value problem. In order to study the biharmonic problem, various boundary conditions are adopted, e.g., the Dirichlet problem [17, 19], the Neumann problem [9], the mixed problem [6, 8], the Navier boundary conditions [13], the Riquier-Neumann boundary conditions, the Robin boundary value problems [4] and the supported boundary condition, see [14, 10, 12].

These represent an important class of inverse problems known to be generally ill-posed, in which the existence, uniqueness, and stability of their solutions are not always guaranteed (see, e.g., [5, 1, 11, 13, 16, 4]). In many experimental cases, the boundary conditions of the considered problem domain are partially or entirely unknown, which cannot be measured because of physical difficulties or geometrical inaccessibility. Here, direct methods are complicated to apply.

Regarding the biharmonic equation, it should be pointed out that the presence of points where the type of boundary conditions change usually leads to local singularities in the solution [14]. This surely puts us in the context of identification problems related to the detection of unknown boundaries, which are essential in many engineering applications, such as detection of corrosion [6], determining the surface of a submarine [2], detecting the boundary of cracks [3, 6]. Our work will focus on the biharmonic equation with several boundary conditions to find a set on which the solution and a specific combination of its derivatives (or a specific combination of its derivatives) vanish. Here, the existence and the uniqueness will be investigated.

Modeling and problem formulation

Let $\Omega \subset \mathbb{R}^2$, be a bounded doubly-connected domain with piece-wise smooth boundary $\partial\Omega = \Gamma_m \cup \Gamma_c$, where Γ_m and Γ_c are two given curves, smooth and closed of class C^2, by n we denote the outward unit normal to $\partial\Omega$. Let $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ be a solution of the following biharmonic equation:

$$\boldsymbol{\Delta^2 u = 0}, \quad \boldsymbol{in\ \Omega} \tag{1.1}$$

that is equivalent to system of equations [4]:

$$\begin{cases} \boldsymbol{\Delta u = w}, & \boldsymbol{in\ \Omega} \\ \boldsymbol{\Delta w = 0}, & \boldsymbol{in\ \Omega} \end{cases} \tag{1.2}$$

with mixed boundary conditions given on the accessible part Γ_m of the boundary by:

$$\begin{cases} \boldsymbol{u} & \boldsymbol{= u_0}, & \boldsymbol{on\ \Gamma_m} \\ \boldsymbol{\frac{\partial u}{\partial n}} & \boldsymbol{= u_1}, & \boldsymbol{on\ \Gamma_m} \\ \boldsymbol{w} & \boldsymbol{= u_2}, & \boldsymbol{on\ \Gamma_m} \\ \boldsymbol{\frac{\partial w}{\partial n}} & \boldsymbol{= u_3}, & \boldsymbol{on\ \Gamma_m} \end{cases} \tag{1.3}$$

Where $\frac{\partial}{\partial n}$ denote the outward unit normal to $\partial\Omega$. We assume that Γ_m is an internal curve and Γ_c is an external curve, both have polar coordinates representations of the form $r = f(\theta)$, where f is a differentiable function and 2π-periodic.

This mathematical model is well known in 2D stokes flows, and in elasticity, where the functions u and w represent the stream function and vorticity in Stokes flows, whilst they represent the deflection and bending moment in elasticity [5, 9].

In [16], it is presented that, if u and its normal derivative $\frac{\partial u}{\partial n}$ or u and w or u and $\frac{\partial w}{\partial n}$ are prescribed at all points of the boundary $\partial\Omega$, this enables us to determine uniquely the solution u everywhere in its definition domain Ω, then it is well-posed direct problem [6, 8]. However, in the practice it is not always possible to specify the boundary conditions at all points on the boundary of the considered domain of the solution and some other boundary information may be given elsewhere [16, 7, 4]. In this case, the problem is called an inverse problem for the biharmonic equation which is ill-posed.

In what follows, we assume that Γ_m is known, and Γ_c is unknown. Our approach considers the situation where boundary conditions u, $\frac{\partial u}{\partial n}$, Δu, $\frac{\partial \Delta u}{\partial n}$ are given on the part Γ_m of the boundary, and the unknown part Γ_c is assumed to have an additional information. The problem (1.1)-(1.3) admits a unique solution u for a compatible data on the part Γ_m of the boundary [8].

The inverse problem we are consider with is: given Γ_m, u_0, u_1, u_2, and u_3. Find the shape Γ_c such as one of the following boundary conditions is satisfy:

$$\boldsymbol{u = \frac{\partial u}{\partial n} = 0}, \quad \boldsymbol{on\ \Gamma_c} \tag{1.4}$$

Or

$$\boldsymbol{u = w = 0}, \quad \boldsymbol{on\ \Gamma_c} \tag{1.5}$$

Or

$$\boldsymbol{\frac{\partial u}{\partial n} = w = 0}, \quad \boldsymbol{on\ \Gamma_c} \tag{1.6}$$

Which correspond to the homogeneous Dirichlet boundary condition, homogeneous Navier boundary condition also called (simply supported constraint boundary condition), and the homogeneous mixed boundary condition, respectively (see [9, 13, 14, 10, 12]).

This inverse problem is usually encountered in elastic plates, for example, finding cracks in a medium from measurements of an elastic field on a surface of the medium [3, 6, 14].

Many articles are devoted to detecting a boundary for the problem (1.1)-(1.3). For example, in [3], it is presented a method to find an unknown boundary for the biharmonic equation in the half open plane by using the Fourier transformation of data. Recently, in [6], the fundamental solution in combination with the minimization method is used to determine an unknown sub-boundary. However, the existence and the uniqueness were not investigated.

It is known that [3, 6], if we just have $u = 0$ on Γ_c, and in this case it is necessary to determine a perfectly conductive boundary crack Γ_c. But the uniqueness of Γ_c cannot be guaranteed. The significance of our investigation is to prescribe the solution u and their normal derivative

on Γ_c as indicated in (1.4), or u and w on Γ_c as indicated in (1.5), or the normal derivative of u and w as indicated in (1.6). In this case, the uniqueness of Γ_c is guaranteed under conditions assumed on the available data (see the rest of this article).
Theoretical results are standard and well known for the Laplace case (see [23]). On the other hand, in [2], the author proposed a simple analytical method based on the Fourier series to find an unknown curve from the extra boundary conditions, in addition, he has used the properties of the harmonic function to study the existence and uniqueness. We follow this work and we extend the techniques to (1.1)-(1.3)-(1.4)-(1.5)-(1.6).
For the outline of this paper, in the second part, we consider the polar coordinate representation of the solution to the biharmonic equation given in [18], and assume that the available Cauchy data have a Fourier expansion. Matching the boundary conditions leads to a linear system equations for coefficients to be determined. In the third section, we briefly discuss the open issue of existence and uniqueness, and we conclude with analytical examples to show the feasibility of this study.

Preliminary and basic results

It is advantageous to formulate elliptic problems in the plane in terms of polar coordinates [5, 6]. So, the following variable change, $x = r cos\theta$, $y = r sin\theta$ are naturally introduced on the problem (1.1)-(1.3), where the radial coordinate is denoted by r and the angular coordinate is denoted by θ. Therefore, the system of equations (1.2) in polar coordinates representation is given as:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = w \tag{2.1}$$

$$\frac{\partial^4 u}{\partial r^4} + \frac{2}{r}\frac{\partial^3 u}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^3}\frac{\partial u}{\partial r} - \frac{2}{r^3}\frac{\partial^3 u}{\partial r\partial \theta^2} + \frac{2}{r^2}\frac{\partial^4 u}{\partial r^2\partial \theta^2} + \frac{4}{r^4}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^4}\frac{\partial^4 u}{\partial \theta^4} = 0 \tag{2.2}$$

In 2D problems, the biharmonic equation can be solved by a repeated application of variables separation procedures [18]. Therefore, the complete general solution of the biharmonic equation applicable to the plane elastic in a doubly connected domain was given by Michell (1863 -1940) [5], as follows:

$$\begin{aligned} u(r,\theta) \quad &= c_{0,1} + c_{0,2}r^2 + c_{0,3}\ln(r) + c_{0,4}r^2\ln(r) + d_{0,3}\theta + d_{0,4}r^2\theta \\ &+ \left(\frac{c_{1,1}}{r} + c_{1,2}r + c_{1,3}r^3 + c_{1,4}(r\ln r) + cr\theta\right)\cos(\theta) \\ &+ \left(\frac{d_{1,1}}{r} + d_{1,2}r + d_{1,3}r^3 + d_{1,4}(r\ln r) + dr\theta\right)\sin(\theta) \\ &+ \sum_{n=2}^{\infty} \left(c_{n,1}r^{-n} + c_{n,2}r^n + c_{n,3}r^{2-n} + c_{n,4}r^{2+n}\right)\cos(n\theta) \\ &+ \sum_{n=2}^{\infty} \left(d_{n,1}r^{-n} + d_{n,2}r^n + d_{n,3}r^{2-n} + d_{n,4}r^{2+n}\right)\sin(n\theta) \end{aligned} \tag{2.3}$$

where $c, d,\ and\ c_{n,1}, c_{n,2}, c_{n,3}, c_{n,4}, d_{n,1}, d_{n,2}, d_{n,3}, d_{n,4},\ for\ n \in \mathbb{N}$, are unknown coefficients to be determined uniquely from the uniqueness of the solution of the Cauchy problem (1.1)-(1.3), and by satisfying the boundary conditions [8, 15, 5].
Without loss of generality, we assume that Γ_m is the unit circle and Γ_c is a curve that contain Γ_m. The functions $u_0(\theta)$, $u_1(\theta)$, $u_2(\theta)$, $u_3(\theta)$ are assumed to be L^2 integrable on the interval $[0,2\pi]$. Hence, all of them admit development in terms of the Fourier expansion [2, 17]

$$\begin{aligned} \mathbf{u_0} &= \mathbf{A_0} + \textstyle\sum_{n=1}^{\infty} \mathbf{A_n}\cos(n\theta) + \mathbf{B_n}\sin(n\theta) \\ \mathbf{u_1} &= \mathbf{A_0}' + \textstyle\sum_{n=1}^{\infty} \mathbf{A_n}'\cos(n\theta) + \mathbf{B_n}'\sin(n\theta) \\ \mathbf{u_2} &= \mathbf{A''_0} + \textstyle\sum_{n=1}^{\infty} \mathbf{A_n}''\cos(n\theta) + \mathbf{B_n}''\sin(n\theta) \\ \mathbf{u_3} &= \mathbf{A'''_0} + \textstyle\sum_{n=1}^{\infty} \mathbf{A_n}'''\cos(n\theta) + \mathbf{B_n}'''\sin(n\theta) \end{aligned} \tag{2.4}$$

In order to compute the coefficients, we restrict (2.3) to the part Γ_m of the boundary $\partial\Omega$ requiring that $u|_{\Gamma_m} = u_0$, $\frac{\partial u}{\partial n}|_{\Gamma_m} = u_1$, $\Delta u|_{\Gamma_m} = u_2$, $\frac{\partial \Delta u}{\partial n}|_{\Gamma_m} = u_3$, and by matching the boundary conditions (2.4) one can obtain that:

$$\begin{cases} c_{0,1}+c_{0,2}=A_0 \\ c_{1,1}+c_{1,2}+c_{1,3}=A_1 \\ c_{n,1}+c_{n,2}+c_{n,3}+c_{n,4}=A_n & n\geq 2 \\ d_{1,1}+d_{1,2}+d_{1,3}=B_1 \\ d_{n,1}+d_{n,2}+d_{n,3}+d_{n,4}=B_n & n\geq 2 \end{cases}, \quad (2.5)$$

$$\begin{cases} 2c_{0,2}+c_{0,3}+c_{0,4}=A_0{}' \\ -c_{1,1}+c_{1,2}+3c_{1,3}+c_{1,4}=A_1{}' \\ -nc_{n,1}+nc_{n,2}+(2-n)c_{n,3}+(2+n)c_{n,4}=A_n{}' & n\geq 2 \\ -d_{1,1}+d_{1,2}+3d_{1,3}+d_{1,4}=B_1{}' \\ -nd_{n,1}+nd_{n,2}+(2-n)d_{n,3}+(2+n)d_{n,4}=B_n{}' & n\geq 2 \end{cases}, \quad (2.6)$$

$$\begin{cases} 4c_{0,2}+4c_{0,4}=A_0{}'' \\ 8c_{1,3}+2c_{1,4}=A_1{}'' \\ (4-4n)c_{n,3}+(4+4n)c_{n,4}=A_n{}'' & n\geq 2 \\ 8d_{1,3}+2d_{1,4}=B_1{}'' \\ (4-4n)d_{n,3}+(4+4n)d_{n,4}=B_n{}'' & n\geq 2 \end{cases} \quad (2.7)$$

$$\begin{cases} 4c_{0,4}=A_0{}''' \\ 8c_{1,3}-2c_{1,4}=A_1{}''' \\ (4n^2-4n)c_{n,3}+(4n^2+4n)c_{n,4}=A_n{}''' & n\geq 2 \\ 8d_{1,3}-2d_{1,4}=B_1{}''' \\ (4n^2-4n)d_{n,3}+(4n^2+4n)d_{n,4}=B_n{}''' & n\geq 2 \end{cases}, \quad (2.8)$$

and $c=d=d_{0,1}=d_{0,2}=d_{0,3}=d_{0,4}=0$.
By solving the systems (2.5)-(2.6)-(2.7)-(2.8) then we obtained that:

$$\begin{array}{ll} c_{0,1}=A_0+\frac{1}{4}(A_0'''-A_0''), & d_{0,1}=d_{0,2}=d_{0,3}=d_{0,4}=0 \\ c_{0,2}=\frac{1}{4}(A_0''-A_0'''), & c=0 \\ c_{0,3}=A_0'+\frac{1}{4}(A_0'''-2A_0'') & d=0 \\ c_{0,4}=\frac{1}{4}A_0''' & \\ c_{1,1}=\frac{1}{2}A_1-\frac{1}{2}A'_1+\frac{1}{16}(3A_1''-A_1'''), & d_{1,1}=\frac{1}{2}B_1-\frac{1}{2}B'_1+\frac{1}{16}(3B_1''-B_1''') \\ c_{1,2}=\frac{1}{2}A_1+\frac{1}{2}A'_1-\frac{1}{4}A_1'', & d_{1,2}=\frac{1}{2}B_1+\frac{1}{2}B'_1-\frac{1}{4}B_1'' \\ c_{1,3}=\frac{1}{16}(A_1''+A_1'''), & d_{1,3}=\frac{1}{16}(B_1''+B_1''') \\ c_{1,4}=\frac{1}{4}(A_1''-A_1'''), & d_{1,4}=\frac{1}{4}(B_1''-B_1''') \\ c_{n,1}=\frac{1}{2}A_n-\frac{1}{2n}A_n{}'+\frac{1-n}{n}c_{n,3}+\frac{1}{n}c_{n,4}, & d_{n,1}=\frac{1}{2}B_n-\frac{1}{2n}B_n{}'+\frac{1-n}{n}d_{n,3}+\frac{1}{n}d_{n,4} \\ c_{n,2}=\frac{1}{2}A_n+\frac{1}{2n}A_n{}'-\frac{1}{n}c_{n,3}-\frac{n+1}{n}c_{n,4}, & d_{n,2}=\frac{1}{2}B_n+\frac{1}{2n}B_n{}'-\frac{1}{n}d_{n,3}-\frac{n+1}{n}d_{n,4} \\ c_{n,3}=\frac{1}{8n^2-8n}(A_n'''-nA_n''), & d_{n,3}=\frac{1}{8n^2-8n}(B_n'''-nB_n'') \\ c_{n,4}=\frac{1}{8n^2+8n}(nA_n''+A_n'''), & d_{n,4}=\frac{1}{8n^2+8n}(nB_n''+B_n''') \end{array} \quad (2.9)$$

Therefore, the solution of the boundary value problem (1.1)-(1.3) in the region between Γ_m and Γ_c is given as follows:

$$\begin{aligned} u(r,\theta) &= c_{0,1}+c_{0,2}r^2+c_{0,3}\ln(r)+c_{0,4}r^2\ln(r) \\ &+\left(\frac{c_{1,1}}{r}+c_{1,2}r+c_{1,3}r^3+c_{1,4}r\ln r\right)\cos(\theta) \\ &+\left(\frac{d_{1,1}}{r}+d_{1,2}r+d_{1,3}r^3+d_{1,4}r\ln r\right)\sin(\theta) \\ &+\sum_{n=2}^{\infty}\left(c_{n,1}r^{-n}+c_{n,2}r^n+c_{n,3}r^{2-n}+c_{n,4}r^{2+n}\right)\cos(n\theta) \\ &+\sum_{n=2}^{\infty}\left(d_{n,1}r^{-n}+d_{n,2}r^n+d_{n,3}r^{2-n}+d_{n,4}r^{2+n}\right)\sin(n\theta) \end{aligned} \quad (2.10)$$

It should be pointed out that, if $u_2 = u_3 = 0$, then by substituting in (2.4) and (2.9) one can obtain that:

$$u(r,\theta) = c_{0,1} + c_{0,3}\ln(r) + \sum_{n=1}^{\infty} \left(c_{n,1}r^{-n} + c_{n,2}r^{n}\right)\cos(n\theta) + \left(d_{n,1}r^{-n} + d_{n,2}r^{n}\right)\sin(n\theta)$$

which corresponds to the solution of the harmonic equation in a doubly connected domain.

Existence and uniqueness

Taking a trivial case where, $u_0 = u_1 = u_2 = u_3 = 0$, and from (2.9), therefore $u \equiv 0$ in Ω. That means there exists an infinite number of portions Γ_c which satisfy the equations (1.4)-(1.5)-(1.6). Therefore, we can confine ourselves in a favorable situation by assuming that Γ_m is known, and $|u_0| + |u_1| + |u_2| + |u_3| \neq 0$,i.e. at least one of the available data does not vanish identically.

In order to examine the existence and uniqueness of the non-accessible part Γ_c, it is necessary to examine each of the equations (1.4), (1.5) and (1.6).

The existence

If (1.4) is satisfy, then, $u = \frac{\partial u}{\partial n} = 0$ on Γ_c and $u_0 = u_1 = 0$, i.e., $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, therefore, $u = 0$ in Ω and $u_0 = u_1 = u_2 = u_3 = 0$, however, this provides a contradiction if $u_2 \neq 0$ and $u_0 = u_1 = 0$, for example.

If (1.5) is satisfy, then, $u = w = 0$ on Γ_c and $u_0 = u_2 = 0$, i.e., $u = w = 0$ on $\partial\Omega$. The maximum-minimum principle for harmonic functions implies that $w = 0$ in Ω, then, $\Delta u = 0$ in Ω and $u = 0$ on $\partial\Omega$, therefore, $u = 0$ in Ω and $u_0 = u_1 = u_2 = u_3 = 0$ however, this provides a contradiction if $u_1 \neq 0$ and $u_0 = u_2 = 0$, for example.

If (1.6) is satisfy, then, $\frac{\partial u}{\partial n} = w = 0$ on Γ_c and $u_1 = u_2 = 0$, i.e., $\frac{\partial u}{\partial n} = w = 0$ on $\partial\Omega$. We have already obtained that $\Delta u = 0$ in Ω, and satisfy $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Thus, $u = constant$ in Ω and consequently $u_0 = constant$ and $u_1 = u_2 = u_3 = 0$. However, this contradicts if $u_0 \neq constant$ and $u_1 = u_2 = 0$, for example. From the above discussion, the following remark can be cited:

Theorem 3.1: *The existence of a solution to the inverse problem (1.1)-(1.3)-(1.4)-(1.5)-(1.6) cannot be guaranteed for arbitrary data u_0, u_1, u_2, u_3.*

The uniqueness

The uniqueness of solution to (1.1)-(1.3)-(1.4) is guaranteed, let Γ_c, $\Gamma_c{}'$ two separate solutions, then, there exist Ω' a domain bounded by certain parts of Γ_c and $\Gamma_c{}'$, in which there exist a biharmonic function, u, verify $\Delta^2 u = 0$ in Ω' and $u = \frac{\partial u}{\partial n} = 0$ on the boundary of Ω', then $u = 0$ in Ω'. This and the unique continuation property for biharmonic functions [20] imply that $u = 0$ in it's definition domain Ω and $u_0 = u_1 = u_2 = u_3 = 0$. However, this contradicts our assumption. (see example 3.1). we can state the following result:

Theorem 3.2: *Assume that in (1.1)-(1.4)-(1.5)-(1.6) we have $u = \frac{\partial u}{\partial n} = 0$ on Γ_c, then the data u_0, u_1, u_2, u_3 uniquely determine Γ_c provided that, $|u_0| + |u_1| + |u_2| + |u_3| \neq 0$.*

The uniqueness of solution to (1.2)-(1.3)-(1.5) is guaranteed, let Γ_c, $\Gamma_c{}'$ two separate solutions then, there exist Ω' a domain bounded by certain parts of Γ_c and $\Gamma_c{}'$, in which there exist a biharmonic function ,u, verify $\Delta^2 u = \Delta w = 0$ in Ω' and satisfy $u = w = 0$ on the boundary of Ω', now, the maximum-minimum principle for harmonic functions implies that $w = 0$ in Ω' and $\Delta u = 0$ in Ω', then, $u = 0$ in Ω'. Thus, by the unique continuation property for harmonic functions [20] we obtain that $u = 0$ in it's definition domain Ω, and $u_0 = u_1 = u_2 = u_3 = 0$. However, this contradicts our assumption. (see example 3.2). We have the following result:

Theorem 3.3: *Suppose that in (1.2)-(1.4)-(1.5)-(1.6) we have $u = w = 0$ on Γ_c, then $u|_{\Gamma_m} = u_0$, $\frac{\partial u}{\partial n}|_{\Gamma_m} = u_1$, $w|_{\Gamma_m} = u_2$ and $\frac{\partial w}{\partial n}|_{\Gamma_m} = u_3$ uniquely determine Γ_c provided that, $|u_0| + |u_1| + |u_2| + |u_3| \neq 0$.*

For the uniqueness of the solution to (1.2)-(1.3)-(1.6) , let Γ_c, $\Gamma_c{}'$ two separate solutions, then, there exist Ω' a domain bounded by certain parts of Γ_c and $\Gamma_c{}'$, in which there exist a biharmonic function, u, verify $\Delta^2 u = \Delta w = 0$ in Ω' and satisfy $\frac{\partial u}{\partial n} = w = 0$ on the boundary of Ω', therefore, $w = 0$ in Ω', and $\Delta u = 0$ in Ω' and satisfy $\frac{\partial u}{\partial n} = 0$ on the boundary of Ω', therefore, $u = constant$ in Ω' and based on the unique property of continuity of an elliptical function we found $u = constant$ in it's definition domain Ω, thus, $u_0 = constant$ and $u_1 = u_2 = u_3 = 0$, then, there is at most one solution Γ_c provided that $u_0 \neq constant$ or $|u_1| + |u_2| + |u_3| \neq 0$, (see example 3.3), and we can state the following result:

Theorem 3.4: *Suppose that in (1.2)-(1.4)-(1.5)-(1.6) we have* $\frac{\partial u}{\partial n} = w = 0$ *on* Γ_c, *then* $u|_{\Gamma_m} = u_0$, $\frac{\partial u}{\partial n}|_{\Gamma_m} = u_1$, $w|_{\Gamma_m} = u_2$ *and* $\frac{\partial w}{\partial n}|_{\Gamma_m} = u_3$ *uniquely determine* Γ_c *provided that* $u_0 \neq constant$ *or* $|u_1| + |u_2| + |u_3| \neq 0$.

Determination of Γ_c

Suppose that $r = f(\theta)$ is a representation of Γ_c. From (1.4), (1.5) and (1.6), we can find the unknown function $f(\theta)$ numerically, by solving the following systems of equations:

$$\begin{cases} \boldsymbol{u(f(\theta),\theta) = 0} \\ \boldsymbol{\frac{\partial u}{\partial n}(f(\theta),\theta) = 0} \end{cases} \tag{3.1}$$

if condition (1.4) is considered, and

$$\begin{cases} \boldsymbol{u(f(\theta),\theta) = 0} \\ \boldsymbol{w(f(\theta),\theta) = 0}' \end{cases} \tag{3.2}$$

if condition (1.5) is considered, and

$$\begin{cases} \boldsymbol{\frac{\partial u}{\partial n}(f(\theta),\theta) = 0} \\ \boldsymbol{w(f(\theta),\theta) = 0} \end{cases}, \tag{3.3}$$

if condition (1.6) is considered, by applying the formula

$$\frac{\partial u}{\partial n}\Big|_{r=f(\theta)} = \frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial u}{\partial \theta}\frac{\partial f}{\partial \theta}\Big|_{r=f(\theta)},$$

where

$$\nabla u = \frac{\partial u}{\partial r}e_r + \frac{1}{r}\frac{\partial u}{\partial \theta}e_\theta, \qquad \text{and } n = e_r - \frac{1}{r}f'(\theta)e_\theta$$

and the vectors e_r and e_θ are unit vectors in polar coordinates.

Numerical examples

In what follows, we consider that Γ_m is the unit circle and we wish to find Γ_c. Here, we give the data u_0, u_1, u_2, u_3 on Γ_m, and we compute the coefficients from (2.9). Then, by substituting in (2.10), one obtains the solution $u(r,\theta)$ of the problem (1.1)-(1.3).

This inverse problem is ill-posed and its numerical solution is difficult (see also [2]). For some simple cases, we try to describe (3.1), (3.2) and (3.3) as a transcendental equations, which can be solved analytically by using the inverse functions.

Example 3.1: *Let* $u_0 = -2 - e^2$, $u_1 = -2 + 2e^2$, $u_2 = -4$, $u_3 = 0$. *From (2.9) we obtain:* $c_{0,1} = -e^2$, $c_{0,2} = -1$, $c_{0,3} = 2e^2$, $c_{0,4} = 0$. *Therefore :* $u = -e^2 - r^2 + 2e^2 lnr$.

Equations (1.4) take the forms $-e^2 - r^2 + 2e^2\ln r = 0$ and $-2r + \frac{2e^2}{r} = 0$, then Γ_c is a circle of radius e as shown in Figure 1.

Equations (1.5) take the forms $-e^2 - r^2 + 2e^2\ln r = 0$ and $\Delta u = -4 \neq 0$. They have no common solutions.

Equations (1.6) take the forms $-2r + \frac{2e^2}{r} = 0$ and $\Delta u = -4 \neq 0$. They have no common solutions.

Example 3.2: *Let* $u_0 = -2$, $u_1 = -3 - e^2$, $u_2 = 4$, $u_3 = -4$. *According to (2.9) we obtain* $c_{0,1} = 0$, $c_{0,2} = 2$, $c_{0,3} = -e^2$, $c_{0,4} = -1$. *Therefore* $u = 2r^2 - e^2 lnr - r^2 lnr$.

The equations (1.4), become $2r^2 - e^2\ln r - r^2\ln r = 0$ and $3r - \frac{e^2}{r} - 2r\ln r = 0$. They have no common solutions.

The equations (1.5), take the forms $2r^2 - e^2\ln r - r^2\ln r = 0$, $4 - 4\ln r = 0$, then Γ_c is a circle of radius e as shown in Figure 1.

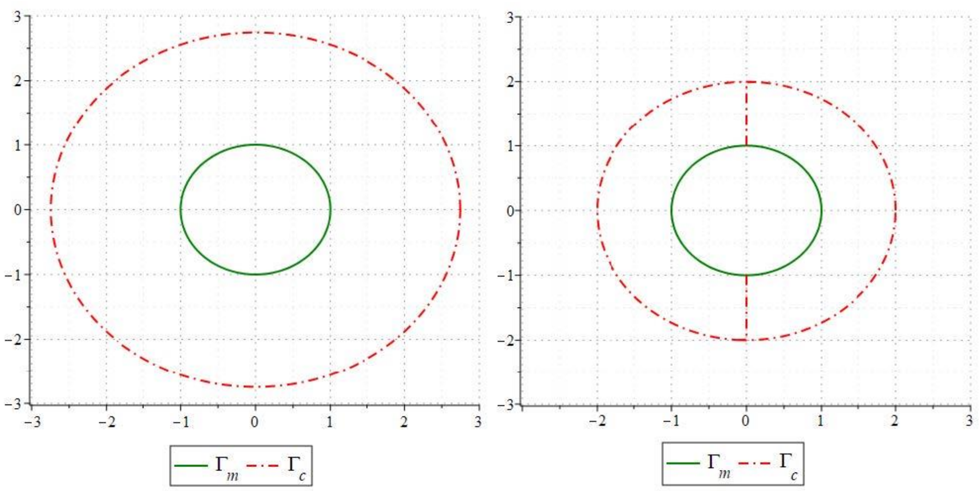
The equations (1.6) become $3r - \frac{e^2}{r} - 2r\ln r = 0$ and $4 - 4\ln r = 0$. They have no common solutions.

Example 3.3: *Let* $u_0 = (\frac{21}{4} + 4ln2)cos(\theta)$, $u_1 = (\frac{7}{4} + 4ln2)cos(\theta)$, $u_2 = -6$, $u_3 = 10$. *According to (2.9) we obtain:* $c_{1,1} = 0$, $c_{1,2} = 1 + 4ln2$, $c_{1,3} = \frac{1}{4}$, $c_{1,4} = -4$. *Therefore:* $u = \left[(1 + 4ln2)r + \frac{r^3}{4} - 4rlnr\right] cos(\theta)$.

The equations (1.4), become $\left[(1 + 4\ln 2)r + \frac{r^3}{4} - 4r\ln r\right] \cos(\theta) = 0$ and $(-3 + 4\ln 2 + \frac{3r^2}{4} - 4\ln r)\cos(\theta) = 0$. They have no common solutions.

The equations (1.5), take the forms $\left[(1 + 4\ln 2)r + \frac{r^3}{4} - 4r\ln r\right] \cos(\theta) = 0$ and $(2r - \frac{8}{r})\cos\theta = 0$. They have no common solutions.

The equations (1.6) become $(-3 + 4\ln 2 + \frac{3r^2}{4} - 4\ln r)\cos(\theta) = 0$ and $(2r - \frac{8}{r})\cos\theta = 0$ then Γ_c is the circle of radius 2 union the dash-dotted line in the annular space. as shown in Figure 1b.



(a) For the examples 3.1 and 3.2. (b) For the example 3.3.

Figure 1: The geometric shape of boundary Γ_c.

CONCLUSION

We investigated a geometrical inverse problem for the biharmonic equation in the doubly connected domain of the plane, where the uniqueness of the solution of Cauchy's problem is ascertained. Then, by measuring the Cauchy data over a specific surface, we can uniquely determine the solution everywhere in its domain of definition. In particular, find sets on which the solution and a certain combination of its derivatives (or a specific combination of its derivatives) vanish.

As we are interested in inverse problems, we noted that our method of determining an unknown boundary is purely analytical, consisting in giving a simple solution to the problem and discussing the existence and uniqueness issue. In addition, we considered the ill-posed problem in a smooth boundary. Nevertheless, it should be mentioned that it is possible to extend the idea to a more complex region, as well as to develop a computational method to solve the inverse problem.

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