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Nonlinear Velocity and Shape Control of a Rotating Smart Flexible Beam-like Structure

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ABSTRACT
Smart materials and structures have a great appeal within the aerospace and automotive communities because they promise to enable better performance and functionality over existing structural and functional materials. The idea proposed in this work is part of this challenging and current scenario and can be applied to the study of morphing wings, helicopter blades, etc. Moreover, the mathematical modeling and velocity and shape control of a rotating flexible beam-like structure are investigated. The nonlinear partial differential governing equations of motion are derived using the extended Hamilton's Principle and numerically integrated using a combination of finite difference and the fourth-order Runge–Kutta methods. In order to force the flexible structure to assume the desired shape and simultaneously control the velocity of the rotating axis, the optimal nonlinear control method named state-dependent Riccati equation (SDRE) is considered. This control technique is applied to piezoelectric actuators along the beam and an external torque coming from a DC motor and acting in the rotating axis. The numerical simulation results show that the proposed control technique is efficient when acting along the rotating beam to deform it into a desired shape while also acting on the motor axis to keep the rotation speed constant.

KEYWORDS: Rotating Flexible Structures, Nonlinear Systems, Piezoelectric Actuators, Smart Structures, Shape Control, Nonlinear Control, SDRE Control.

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Authors' contributions
The participation of each author corresponds to the criteria of authorship and contributorship emphasized in the Recommendations for the Conduct, Reporting, Editing, and Publication of Scholarly work in Medical Journals of the International Committee of Medical Journal Editors. Indeed, all the authors have actively participated in the redaction, the revision of the manuscript, and provided approval for this final revised version.

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1. INTRODUCTION
Some types of aircraft change shape to adapt to different external conditions during flight called morphing aircraft [1]. Basically, this type uses the concept of adaptive geometry wings [2]. In the last decades, many works have been published on this subject, for example, [1-5].
The development of intelligent materials (or structures) has received a lot of attention in the field of adaptive geometry aircraft. These new materials enable distributed actuation while simultaneously performing a structural function and significantly reducing the system's weight as a whole (compared to the traditional hydraulic or pneumatic actuation model). Examples of smart materials are as follows: piezoelectric materials, shape memory alloys, and magnetorheological and electro rheological fluids.
The mathematical modeling, dynamics, and control of rotating flexible beam-like continuous structures as a representative model are topics of ongoing research in aerospace, naval, and oceanic engineering fields. In order to actuate along these structures, specific devices such as piezoelectric actuators must be used [6, 7]. The piezoelectric ceramic is considered a type of intelligent material [8]. The piezoelectric elements adhered to the surface of the structure do not add significant weight to the original structure and do not affect its flexibility characteristic.
SDRE is a nonlinear suboptimal control technique that synthesizes a nonlinear feedback control law and allows nonlinearities in the system states while additionally offering great design flexibility through state-dependent weighting matrices [9, 10]. The nonlinear system is not linearized, and the state-dependent matrices are calculated point to point in the state space (or in each integration time step).
The primary purpose of this work is to investigate the use of piezoelectric material to shape a rotating flexible beam-like structure based on the Euler–Bernoulli beam theory into a specific form while keeping the velocity of the rotating axis constant.
A discussion about the contributions of this research is presented in the following.
In the literature, most mathematical models involving distributed parameter systems present some approximation through normal modes and/or shape functions (finite elements) rather than the direct

numerical integration of partial differential equations [11-13]. Approximation of this kind requires the system under analysis to be linear (or, at best, weakly nonlinear). The problem addressed here is allowed to be strongly nonlinear (due to high angular velocities) and the nonlinear partial differential equations are directly integrated using the finite difference method.

Given what was presented in the previous paragraph, the control laws generally found in the literature and associated with smart structures are linear (pneumatic actuators, PI, PD, PID, LQR, modal control, etc.) [12, 14]. In a few cases, nonlinear techniques, such as genetic algorithm or adaptive control, are found. When applied to nonlinear systems such as those discussed in this work, the nonlinear optimal control technique called SDRE produces excellent results—as presented in this article—but it is not commonly found in the literature.

In general, in the literature, piezoelectric actuators are associated with smart flexible structures only for active vibration control [11, 12]. In this work, this type of structure and this type of actuators are used for shape control (the vibration in the structure is eliminated as a by-product of the performance of these actuators).

Most works in the literature focus on static deformation analysis (applying a specific voltage profile to piezoelectric actuators in the open loop). These studies typically concentrate on the optimal placement of piezoelectric actuators [11-16]. This work considers dynamic deformation analysis (applying a specific voltage profile to piezoelectric actuators in a closed loop); that is, a flexible beam-like structure is deformed and keeps this shape in the rotationthatis kept constant also under control.

The rest of this paper is organized as follows: the geometric model of the system investigated in this work is presented in Section 2; the mathematical model of the flexible beam-like structure and the DC motor, comments on the piezoelectric actuator, and the discretization of the mathematical model are presented in Section 3;the nonlinear control method is presented in Section 4; numerical simulations and discussions of results are presented in Section 5, conclusions are presented in Section 6; references are given in Section 7.

2. THE GEOMETRIC MODEL

The geometric model of the system investigated in this work is presented in Figure1.

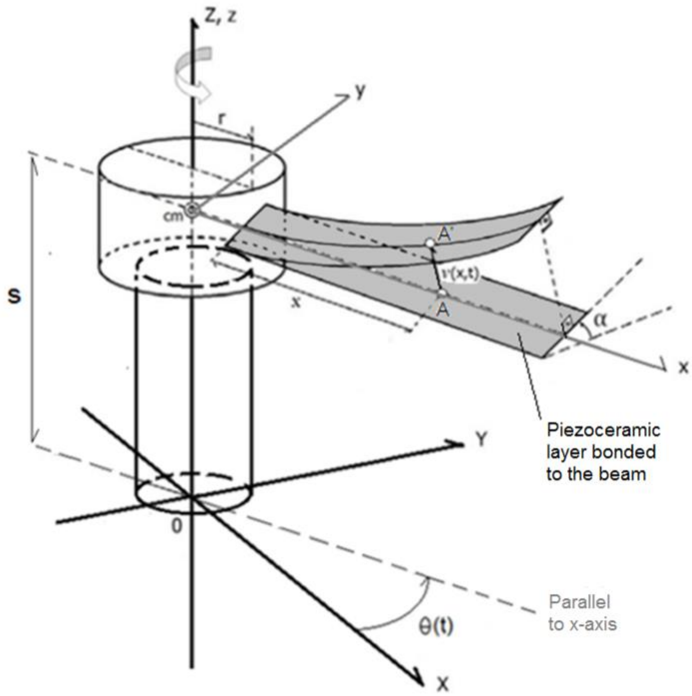

Figure 1. The rotating flexible beam system.

This system comprises a rigid cylindrical body connected to a flexible beam-like structure in rotation about the Z-axis. In this figure, the inertial axis is represented by XYZ, and the moving axis (attached to the system and rotating with it) is represented by xyz.

3. THE MATHEMATICAL MODEL

The governing equations of motion for the system depicted in Figure 1 are obtained through the energy method [17, 18]. This method requires knowledge of the kinetic, potential, and strain energies stored in the system (hub and flexible structure) during the time evolution.

The total kinetic energy, T, of the system is given by the following:

$$T = \frac{1}{2} I_{hub}\dot{\theta}^2 + \frac{1}{2}\int_0^L \rho A \left[\dot{\theta}^2 (r+x)^2 - 2\dot{\theta}(r+x)\dot{v}\sin\alpha + \dot{v}^2 + v^2 \sin^2\alpha\dot{\theta}^2 \right] dx. \tag{1}$$

In Eq. (1) (Fig. 1), $\theta(t)$ is the angular displacement of the hub axis, h represents the vertical constant position of the system, r represents the radius of the hub, α is the angle of rotation of the beam about its longitudinal axis and is a fixed value in this work, *v(x, t)* represents the transversal displacement of the beam, ρ is the density of the material that composes the beam, *A* represents the beam cross-sectional area, and *L* represents the non-deflected length of the beam.

Linear curvature is assumed in this work for the flexible structure [19-21]. The total potential + strain energy, V, of the system is given by the following:

$$V = m_{hub}\, g\, s + \frac{1}{2}\int_0^L \rho A g (s + v\cos\alpha)\, dx + \frac{1}{2}\int_0^L EIv''^2 dx,$$

where *E* represents Young's modulus of the material of which the beam is made and *I* represents the moment of inertia of the cross-sectional area of the beam.

The Lagrangian, L, therefore, is given by the following:

L = T – V,

or

$$L = \frac{1}{2} I_{hub}\dot{\theta}^2 - m_{hub} g s + \frac{1}{2}\int_0^L \rho A \left[\dot{\theta}^2 (r+x)^2 - 2\dot{\theta}(r+x)\dot{v}\sin\alpha + \dot{v}^2 + v^2 \sin^2\alpha\dot{\theta}^2 - g(s + v\cos\alpha) - \frac{EI}{\rho A} v''^2 \right] dx. \quad (2)$$

The torque applied to the axis of the cylinder is not considered at this point, and it will be discussed later when the DC motor equations are introduced. The analysis of the mechanical part of the DC motor provides the governing equation of motion for the variable θ.

The work done by the piezoelectric forces on the beam is given by the following:

$$W_{piezo} = \int_0^L \frac{\partial^2 M_{piezo}(x,t)}{\partial x^2}\, v(x,t)\, dx, \quad (3)$$

where $M_{piezo}(x, t)$ in Eq. (3) is the bending moment applied by the piezoelectric actuator to the beam.

The extended Hamilton's Principle can be given for a mechanical system as follows [17]:

$$\int_{t_1}^{t_2} \left(\delta W + \delta L\right) dt = 0, \quad (4)$$

where W is the total work done by external forces (or loads) on the bodies, t_1 and t_2 are the initial and final times, and L is the Lagrangian of the system.

Substituting Eqs. (2) and (3) into Eq. (4) (where W = W_{piezo}) results in the governing equations of motion for the variable *v*(x, t) as given by the following:

$$\ddot{v} + (r+x)\sin\alpha\ddot{\theta} - v\sin^2\alpha\dot{\theta}^2 + \rho A g\cos\alpha + \frac{EI}{\rho A} v^{iv} = \frac{1}{\rho A} M''_{piezo}, \quad (5)$$

where $M''_{piezo} = \dfrac{\partial^2 M_{piezo}}{\partial x^2}$.

The boundary conditions for the beam can be obtained as a subproduct of the application of the extended Hamilton's Principle and are given to a clamped-free beam by the following:

$$v(0,t) = 0,$$

$$v'(0,t) = 0,$$

$$v''(L,t) = 0,$$

$$v'''(L,t) = 0.$$

The next step in the mathematical modeling of the system depicted in Figure1 is the numerical integration of the partial differential governing equation of motion given by Eq. (5) via the finite difference method.

3.1. About the piezoelectric actuator

A more detailed discussion about these actuators, their mathematical modeling, and incorporation into the governing equations of motion of the rotating flexible structure is not part of the scope of this work. For this work, the idea that adding piezoelectric actuators acting along the flexible structure adds external forces on the right side of Eq. (6) is sufficient.

Equation (5) can also be written as follows [7, 22]:

$$\ddot{v} + (r+x)\sin\alpha\ddot{\theta} + \cos\alpha\ddot{s} - v\sin^2\alpha\dot{\theta}^2 + \rho A g\cos\alpha + \frac{EI}{\rho A} v^{iv} = q_{piezo}(x,t), \quad (6)$$

where $q_{piezo}(x, t)$ in Eq. (6) is the force applied by the piezoelectric actuator to the beam. The external force $q_{piezo}(x, t)$ is also the control force to be applied along the flexible structure.

3.2. Applying the finite difference method

Making explicit the derivatives of v in Eq. (6) results in the following:

$$\frac{\partial^2 v}{\partial t^2} + (r+x)\sin\alpha\ddot{\theta} - v\sin^2\alpha\dot{\theta}^2 + \rho A g\cos\alpha + \frac{EI}{\rho A}\frac{\partial^4 v}{\partial x^4} = \frac{1}{\rho A} q_{piezo}(x,t). \quad (7)$$

The general formula for the central difference using 5 points is given by the following [23]:

$$\frac{\partial^4 v(x_i,\, t)}{\partial x^4} = \frac{v(x_i + 2h,\, t) - 4v(x_i + h,\, t) + 6v(x_i,\, t) - 4v(x_i - h,\, t) + v(x_i - 2h,\, t)}{h^4}. \quad (8)$$

Consider Figure 2.

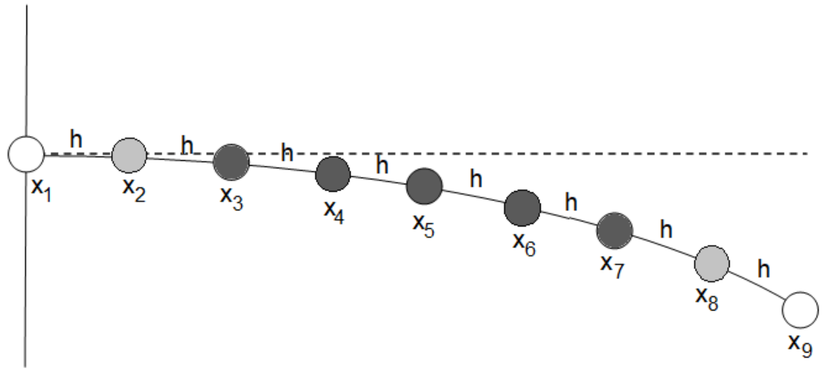

Figure 2. Discretization of a clamped-free beam.

For points 3 to 7, one has, respectively (remembering that $v_1 = 0$), the following:

$$\frac{\partial^4 v(x_3,\ t)}{\partial x^4} = \frac{v_5 - 4v_4 + 6v_3 - 4v_2}{h^4} = \frac{\partial^4 v_3}{\partial x^4}, \tag{9}$$

$$\frac{\partial^4 v(x_4,\ t)}{\partial x^4} = \frac{v_6 - 4v_5 + 6v_4 - 4v_3 + v_2}{h^4} = \frac{\partial^4 v_4}{\partial x^4}, \tag{10}$$

$$\frac{\partial^4 v(x_5,\ t)}{\partial x^4} = \frac{v_7 - 4v_6 + 6v_5 - 4v_4 + v_3}{h^4} = \frac{\partial^4 v_5}{\partial x^4}, \tag{11}$$

$$\frac{\partial^4 v(x_6,\ t)}{\partial x^4} = \frac{v_8 - 4v_7 + 6v_6 - 4v_5 + v_4}{h^4} = \frac{\partial^4 v_6}{\partial x^4}, \tag{12}$$

$$\frac{\partial^4 v(x_7,\ t)}{\partial x^4} = \frac{v_9 - 4v_8 + 6v_7 - 4v_6 + v_5}{h^4} = \frac{\partial^4 v_7}{\partial x^4}. \tag{13}$$

As the structure is clamped at point 1, for this point, one always has the following:

$$v(x_1,\ t) = v_1 = 0. \tag{14}$$

It means that solving an equation for this variable is not necessary.
For point 2, applying the general formula, one has the following:

$$\frac{\partial^4 v(x_2,\ t)}{\partial x^4} = \frac{v(x_2 + 2h,\ t) - 4v(x_2 + h,\ t) + 6v(x_2,\ t) - 4v(x_2 - h,\ t) + v(x_2 - 2h,\ t)}{h^4}. \tag{15}$$

The point $x_2 - 2h$ is outside the beam. However, for point x_1, it is known that $v'(x_1,\ t) = 0$ (since the beam is clamped at this point). Using the central difference formula and considering three points, it can be written that

$$\frac{\partial v(x_1,\ t)}{\partial x} = \frac{v(x_1 + h,\ t) - v(x_1 - h,\ t)}{2h} = \frac{v(x_1 + h,\ t) - v(x_2 - 2h,\ t)}{2h} = 0. \tag{16}$$

Therefore,

$$v(x_1 + h,\ t) = v(x_2 - 2h,\ t). \tag{17}$$

Substituting Eq. (17) into Eq. (15) results in the following:

$$\begin{aligned}\frac{\partial^4 v(x_2,\ t)}{\partial x^4} &= \frac{v(x_2 + 2h,\ t) - 4v(x_2 + h,\ t) + 6v(x_2,\ t) - 4v(x_2 - h,\ t) + v(x_1 + h,\ t)}{h^4} \\ &= \frac{v_4 - 4v_3 + 7v_2 - 4v_1}{h^4} = \frac{v_4 - 4v_3 + 7v_2}{h^4} = \frac{\partial^4 v_2}{\partial x^4}.\end{aligned} \tag{18}$$

For point 8, applying the general formula, one has the following:

$$\frac{\partial^4 v(x_8,\ t)}{\partial x^4} = \frac{v(x_8 + 2h,\ t) - 4v(x_8 + h,\ t) + 6v(x_8,\ t) - 4v(x_8 - h,\ t) + v(x_8 - 2h,\ t)}{h^4}. \tag{19}$$

The point $x_8 + 2h$ is outside the beam. However, for point x_9, it is known that $v''(x_9,\ t) = 0$ (since the beam is free at this point). Using the central difference formula and considering three points, it can be written that

$$\frac{\partial^2 v(x_9,\ t)}{\partial x^2} = \frac{v(x_9 + h,\ t) - 2v(x_9,\ t) + v(x_9 - h,\ t)}{h^2} = 0 \tag{20}$$

or

$$\frac{\partial^2 v(x_9,\ t)}{\partial x^2} = \frac{v(x_8 + 2h,\ t) - 2v(x_9,\ t) + v(x_9 - h,\ t)}{h^2} = 0. \tag{21}$$

Therefore,

$$v(x_8 + 2h,\ t) = 2v(x_9,\ t) - v(x_9 - h,\ t). \tag{22}$$

Substituting Eq. (22) into Eq. (19) results in

$$\frac{\partial^4 \nu(x_8,t)}{\partial x^4} = \frac{2\nu(x_9,t) - \nu(x_9 - h,t) - 4\nu(x_8 + h,t) + 6\nu(x_8,t) - 4\nu(x_8 - h,t) + \nu(x_8 - 2h,t)}{h^4}$$
$$= \frac{-2\nu_9 + 5\nu_8 - 4\nu_7 + \nu_6}{h^4} = \frac{\partial^4 \nu_8}{\partial x^4}. \tag{23}$$

For point 9, applying the general formula, one has the following:

$$\frac{\partial^4 \nu(x_9,\ t)}{\partial x^4} = \frac{\nu(x_9 + 2h,\ t) - 4\nu(x_9 + h,\ t) + 6\nu(x_9,\ t) - 4\nu(x_9 - h,\ t) + \nu(x_9 - 2h,\ t)}{h^4}. \tag{24}$$

The points x_9 + h and x_9 + 2h are outside the beam. However, from Eq. (22), one has the following:

$$\nu(x_8 + 2h,\ t) = 2\nu(x_9,\ t) - \nu(x_9 - h,\ t) = \nu(x_9 + h,\ t). \tag{25}$$

From the problem in question (free end), it is also known that

$$\frac{\partial^3 \nu(x_9,\ t)}{\partial x^3} = 0$$

or using the central difference formula considering 5 points that

$$\frac{\partial^3 \nu(x_9,\ t)}{\partial x^3} = \frac{\nu(x_9 + 2h,\ t) - 2\nu(x_9 + h,\ t) + 2\nu(x_9 - h,\ t) - \nu(x_9 - 2h,\ t)}{2h^3} = 0.$$

Therefore,

$$\nu(x_9 + 2h,\ t) = 2\nu(x_9 + h,\ t) - 2\nu(x_9 - h,\ t) + \nu(x_9 - 2h,\ t). \tag{26}$$

Substituting Eq. (25) into Eq. (26) results in the following:

$$\nu(x_9 + 2h,\ t) = 4\nu(x_9,\ t) - 4\nu(x_9 - h,\ t) + \nu(x_9 - 2h,\ t). \tag{27}$$

Substituting Eq. (25) and Eq. (27) into Eq. (24) results in the following:

$$\frac{\partial^4 \nu(x_9,\ t)}{\partial x^4} = \frac{2\nu(x_9,\ t) - 4\nu(x_9 - h,\ t) + 2\nu(x_9 - 2h,\ t)}{h^4} = \frac{2\nu_9 - 4\nu_8 + 2\nu_7}{h^4} = \frac{\partial^4 \nu_9}{\partial x^4}. \tag{28}$$

So, finally, Eq. (7) can be rewritten for nodes 2 through 9 (considering that x_2 = h, x_3 = 2h, x_4 = 3h,...)as follows:

$$\ddot{v}_2 + (r + h)\sin\alpha_2\ddot{\theta} - \nu_2\sin^2\alpha_2\dot{\theta}^2 + \rho Ag\cos\alpha_2 + \frac{EI}{\rho Ah^4}(\nu_4 - 4\nu_3 + 7\nu_2) = \frac{1}{\rho A}q_{piezo,2}, \tag{29}$$

$$\ddot{v}_3 + (r + 2h)\sin\alpha_3\ddot{\theta} - \nu_3\sin^2\alpha_3\dot{\theta}^2 + \rho Ag\cos\alpha_3 + \frac{EI}{\rho Ah^4}(\nu_5 - 4\nu_4 + 6\nu_3 - 4\nu_2) = \frac{1}{\rho A}q_{piezo,3}, \tag{30}$$

$$\ddot{v}_4 + (r + 3h)\sin\alpha_4\ddot{\theta} - \nu_4\sin^2\alpha_4\dot{\theta}^2 + \rho Ag\cos\alpha_4 + \frac{EI}{\rho Ah^4}(\nu_6 - 4\nu_5 + 6\nu_4 - 4\nu_3 + \nu_2) =$$
$$\frac{1}{\rho A}q_{piezo,4}, \tag{31}$$

$$\ddot{v}_5 + (r + 4h)\sin\alpha_5\ddot{\theta} - \nu_5\sin^2\alpha_5\dot{\theta}^2 + \rho Ag\cos\alpha_5 + \frac{EI}{\rho Ah^4}(\nu_7 - 4\nu_6 + 6\nu_5 - 4\nu_4 + \nu_3) =$$
$$\frac{1}{\rho A}q_{piezo,5}, \tag{32}$$

$$\ddot{v}_6 + (r + 5h)\sin\alpha_6\ddot{\theta} - \nu_6\sin^2\alpha_6\dot{\theta}^2 + \rho Ag\cos\alpha_6 + \frac{EI}{\rho Ah^4}(\nu_8 - 4\nu_7 + 6\nu_6 - 4\nu_5 + \nu_4) =$$
$$\frac{1}{\rho A}q_{piezo,6}, \tag{33}$$

$$\ddot{v}_7 + (r + 6h)\sin\alpha_7\ddot{\theta} - \nu_7\sin^2\alpha_7\dot{\theta}^2 + \rho Ag\cos\alpha_7 + \frac{EI}{\rho Ah^4}(\nu_9 - 4\nu_8 + 6\nu_7 - 4\nu_6 + \nu_5) =$$
$$\frac{1}{\rho A}q_{piezo,7}, \tag{34}$$

$$\ddot{v}_8 + (r + 7h)\sin\alpha_8\ddot{\theta} - \nu_8\sin^2\alpha_8\dot{\theta}^2 + \rho Ag\cos\alpha_8 + \frac{EI}{\rho Ah^4}(-2\nu_9 + 5\nu_8 - 4\nu_7 + \nu_6) = \frac{1}{\rho A}q_{piezo,8}, \tag{35}$$

$$\ddot{v}_9 + (r + 8h)\sin\alpha_9\ddot{\theta} - \nu_9\sin^2\alpha_9\dot{\theta}^2 + \rho Ag\cos\alpha_9 + \frac{EI}{\rho Ah^4}(2\nu_9 - 4\nu_8 + 2\nu_7) = \frac{1}{\rho A}q_{piezo,9}. \tag{36}$$

In Eqs. (29) to (36), the quantity $q_{piezo,\ i}(x_i,\ t)$ with i = 2 to 9, or equivalently $q_{piezo,\ i}(t)$, is the external force on each node that comes from the piezoelectric actuators and is related to the shape control of the structure. The quantities α_i (with i = 2 to 9) represent the different angles for each section of the beam. In this article, all these angles are considered equal.

3.3 The ideal and the nonideal DC motor

The difference between the mathematical models of an ideal and a nonideal dynamical system is given according to the choice of the actuator-structure coupling model; for example, it depends on the way the torque τ is modeled [19, 24].

In case the motor axis is sufficiently short so that one can consider it sufficiently rigid, the total torque, τ, acting on this axis can be written as follows:

$$\tau = -M + I_{axis}\ddot{\theta}. \tag{37}$$

In Eq. (37), M is the flexure moment of the beam, acting on the motor axis. Using this equation, the mechanical equation of the DC motor becomes:

$$(I_{axis}+I_m N_g^2)\ddot{\theta}+(c_m N_g^2)\dot{\theta}-(N_g K_t)\,i_a = M. \tag{38}$$

The momentM can be represented by the assumption of the linear curvature model as follows:

$M = EI v''(x_1, t) = EI\, v_1''(t).$

Using the central difference formula and considering three points, it can be written as follows:

$$\frac{\partial^2 v(x_1, t)}{\partial x^2} = \frac{v(x_1+h, t)-2v(x_1, t)+v(x_1-h, t)}{h^2} = \frac{v_2-2v_1+v(x_1-h, t)}{h^2} = v_1''(t).$$

UsingEqs. (16) and (17),one can write the following:

$v(x_1-h, t) = v(x_2-2h, t) = v(x_1+h, t).$

Therefore,

$$v_1''(t) = \frac{2v_2-2v_1}{h^2}.$$

Finally,

$$M = \frac{2EI}{h^2}(\ v_2 - v_1\ ) = \left(\frac{2EI}{h^2}\right)v_2. \tag{39}$$

The equations of motion for the slewing flexible structure considering an ideal $(\beta=0)$ and a nonideal $(\beta=1)$ power source are given by Eqs. (29) to (36) and by the electrical and mechanical direct current motor governing equations of motion given, respectively, by the following:

$$L_m\frac{di_a}{dt}+R_a i_a + N_g K_b\dot{\theta} = U, \tag{40}$$

$$(I_{axis}+I_m N_g^2)\ddot{\theta}+(c_m N_g^2)\dot{\theta}-(N_g K_t)\,i_a = \beta\left(\frac{2EI}{h^2}\right)v_2. \tag{41}$$

In this work, only the case $\beta=1$ is considered.

The electrical time constant $\frac{L_m}{R_a}$ is often neglected since it is at least one order in magnitude smaller than the mechanical time constant $\frac{I_{axis}+I_m N_g^2}{c_m N_g^2}$. By neglecting $L_m\frac{di_a}{dt}$ in Eq. (40), one has the following:

$$i_a = \frac{U - N_g K_b\dot{\theta}}{R_a}. \tag{42}$$

Substituting Eq. (42) into Eq. (41), the governing equation of the DC motor is simplified as follows:

$$(I_{axis}+I_m N_g^2)\ddot{\theta} + \left(c_m N_g^2 + \frac{N_g^2 K_t K_b}{R_a}\right)\dot{\theta} = \left(\frac{N_g K_t}{R_a}\right)U + \beta\left(\frac{2EI}{h^2}\right)v_2. \tag{43}$$

3.4 The complete set of governing equations of motion

Considering the DC motor equation and the discretized beam equation, the complete set of governing equations of motion is finally given as follows:

$$(I_{axis}+I_m N_g^2)\ddot{\theta} + \left(c_m N_g^2 + \frac{N_g^2 K_t K_b}{R_a}\right)\dot{\theta} = \left(\frac{N_g K_t}{R_a}\right)U + \beta\left(\frac{2EI}{h^2}\right)v_2, \tag{44}$$

$$\ddot{v}_2 + (\ r+h\ )\sin\alpha_2\ddot{\theta} - v_2\sin^2\alpha_2\dot{\theta}^2 + \frac{EI}{\rho A h^4}(v_4-4v_3+7v_2) = \frac{1}{\rho A}q_{piezo,2} - \rho A g\cos\alpha_2, \tag{45}$$

$$\ddot{v}_3 + (\ r+2h\ )\sin\alpha_3\ddot{\theta} - v_3\sin^2\alpha_3\dot{\theta}^2 + \frac{EI}{\rho A h^4}(v_5-4v_4+6v_3-4v_2) = \frac{1}{\rho A}q_{piezo,3} - \rho A g\cos\alpha_3, \tag{46}$$

$$\ddot{v}_4 + (\ r+3h\ )\sin\alpha_4\ddot{\theta} - v_4\sin^2\alpha_4\dot{\theta}^2 + \frac{EI}{\rho A h^4}(v_6-4v_5+6v_4-4v_3+v_2) = \frac{1}{\rho A}q_{piezo,4} - \rho A g\cos\alpha_4, \tag{47}$$

$$\ddot{v}_5 + (\ r+4h\ )\sin\alpha_5\ddot{\theta} - v_5\sin^2\alpha_5\dot{\theta}^2 + \frac{EI}{\rho A h^4}(v_7-4v_6+6v_5-4v_4+v_3) = \frac{1}{\rho A}q_{piezo,5} - \rho A g\cos\alpha_5, \tag{48}$$

$$\ddot{v}_6 + (\ r+5h\ )\sin\alpha_6\ddot{\theta} - v_6\sin^2\alpha_6\dot{\theta}^2 + \frac{EI}{\rho A h^4}(v_8-4v_7+6v_6-4v_5+v_4) = \frac{1}{\rho A}q_{piezo,6} - \rho A g\cos\alpha_6, \tag{49}$$

$$\ddot{v}_7 + (\ r+6h\ )\sin\alpha_7\ddot{\theta} - v_7\sin^2\alpha_7\dot{\theta}^2 + \frac{EI}{\rho A h^4}(v_9-4v_8+6v_7-4v_6+v_5) = \frac{1}{\rho A}q_{piezo,7} - \rho A g\cos\alpha_7, \tag{50}$$

$$\ddot{v}_8+(r+7h)\sin\alpha_8\ddot{\theta}-\nu_8\sin^2\alpha_8\dot{\theta}^2+\frac{EI}{\rho Ah^4}(-2\nu_9+5\nu_8-4\nu_7+\nu_6)=\frac{1}{\rho A}q_{piezo,8}- \tag{51}$$

$\rho Ag\cos\alpha_8$,

$$\ddot{v}_9+(r+8h)\sin\alpha_9\ddot{\theta}-\nu_9\sin^2\alpha_9\dot{\theta}^2+\frac{EI}{\rho Ah^4}(2\nu_9-4\nu_8+2\nu_7)=\frac{1}{\rho A}q_{piezo,9}-\rho Ag\cos\alpha_9. \tag{52}$$

Writing Eqs. (44) to (53) in state space variables resultsin a matrix equation that has the following form:

$$\dot{x}=A(x)\,x+B\,u+C. \tag{53}$$

In Eq. (53), vector **C** contains the gravity terms. In the following numerical simulations, parameter α is assumed to be the same for all the nodes and equal to 90 degrees. In this way, vector **C** vanishes and the equations of motion in state space variables take the form $\dot{\mathbf{x}}=\mathbf{A(x)x}+\mathbf{Bu}$.

4. NONLINEAR CONTROL: STATE-DEPENDENT RICCATI EQUATION

The SDRE method for designing control algorithms for nonlinear systems relies on representing a nonlinear system's dynamics with state-dependent matrices to be inserted into SDRE to generate a feedback law [9, 10]. The basic idea of this method is to design the control law by treating the state-dependent matrices as if they were constant and calculating the corresponding linear quadratic control at each numerical integration time step. All the state variables must be available, and the system must be totally actuated.

The main idea of the SDRE method is to represent the following nonlinear system:

$$\dot{x}=f(x)+B(x)u \tag{54}$$

in the form of

$$\dot{x}=A(x)\,x+B(x)u. \tag{55}$$

The choice of state-dependent matrices A(x) and B(x) is not unique, and different controls will result from different choices of A(x) and B(x). Once a suitable choice for A(x) and B(x) is found (the system is controllable or at least stabilizable), there always exists a control law that makes the closed-loop system asymptotically stable.

The feedback law is then given by the following [9, 10]:

$$u=-R^{-1}(x)\,B^T(x)\,P(x)\,x, \tag{56}$$

where P(x) is obtained from the Riccati equation:

$$P(x)A(x)+A^T(x)P(x)+Q(x)-P(x)B(x)R^{-1}(x)B^T(x)P(x)=0. \tag{57}$$

In Eq. (57), Q(x) and R(x) are design matrices that satisfy the positive definiteness conditions Q(x) > 0 and R(x) > 0. These matrices are selected to give weight to the state and the control, as in the case of a linear system.

In this work, the control feedback law is applied to the voltage *U* (responsible for the angular velocity control) and the piezoelectric forces (responsible for the beam shape control).

5. NUMERICAL SIMULATIONS ANDDISCUSSIONS

The objective of this study is to verify the possibility of maintaining a beam-like flexible structure in constant rotation and, at the same time, using piezoelectric actuators located along its length, causing the structure to deform according to a predefined shape and maintain this shape indefinitely when rotating at sufficiently high speeds.

The parameter values used in the numerical simulations are presented in Table 1.

Table 1. Parameters values used in the numerical simulations.

Parameter	Nomenclature	Value
Mass density (beam)	ρ	2700 kg/m^3
Elastic modulus (beam)	E	0.700*10^{11} Pa
Beam length	L	1.500 m
Beam thickness	h_b	0.0005 m
Beam width	w	0.080 m
Radius of the hub	r	0.100 m
Angle of rotation of the beam about its longitudinal axis	α	90º

The values shown in Table 1 are not special; they are just a set of values used so that a numerical simulation can be presented. These values (beam thickness, beam length, hub inertia, and hub radius) can be changed within certain ranges without issues. Attention must be paid: if the beam is too thick, for example, it makes no physical sense to deform it using piezoelectric actuators; if the beam is too long, other issues will arise. Depending on the values chosen, the accuracy of the results may change, or the system may become unstable for the values of the elements of the Q and R matrices chosen, and new weight matrices must be selected.

The governing equations of motion are numerically integrated in time using the Runge–Kutta method with a time step of 0.0005 s. The numerical integration of the governing equations of motion and the implementation of the control law were fully programmed and solved with GNU Octave.

The weighting matrices (obtained through constant adjustments) used in the numerical simulations are as follows:

$$Q = \begin{bmatrix}
600000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 40000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 10000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 10000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10000 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10000 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 40000 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$\mathrm{R} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.005 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.005 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.005 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.005 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.005 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.005 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.005 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.005
\end{bmatrix}$$

For the same Q and R matrices, a change in the parameters does not cause a significant difference in the results (only in the amplitudes of torque and control forces, naturally). Even when the values in the Q and R matrices are changed, the system response does not change significantly. However, they can get worse.

The proposed nonlinear control is capable of maintaining the angularvelocity of the motor axis at the desired velocity of 10 rad/s, as seen in Figure 3. Angular displacement uses the integral of the desired curve for the angular velocity as a reference.

As shown in Figure 3b, the angular velocity converges to the desired value (the angular displacement, therefore, also follows the integral of the angular velocity curve, as shown in Figure 3a).

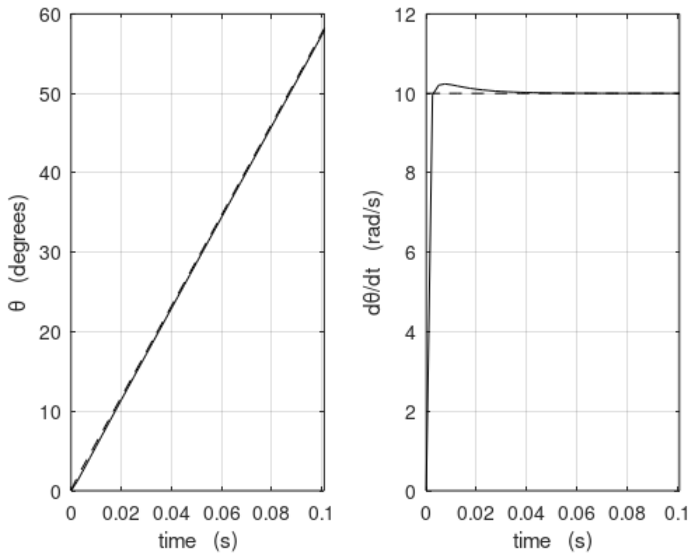

Figure 3. (a) Angular displacement: - - -, reference; —–, numerical result. (b) Angular velocity: - - -, reference; —–, numerical result.

In Figure 4, (a) the control torque generated by the DC motor and (b) the control forces generated at each node by means of the piezoelectric actuators are presented.

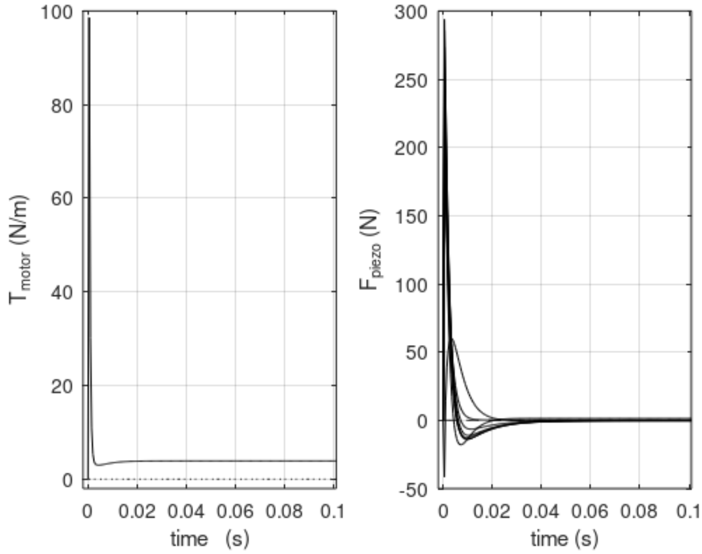

Figure 4. (a) Control torque generated by the direct current motor; (b) Control forces generated on all the piezoelectric actuators.

Figure 5 shows the desired parabolic curve given by v = - 0.285368 x^2 + 0.350719 x to be followed and the curve assumed by the rotating flexible beam-like structure when deformed under the effect of the piezoelectric actuators.

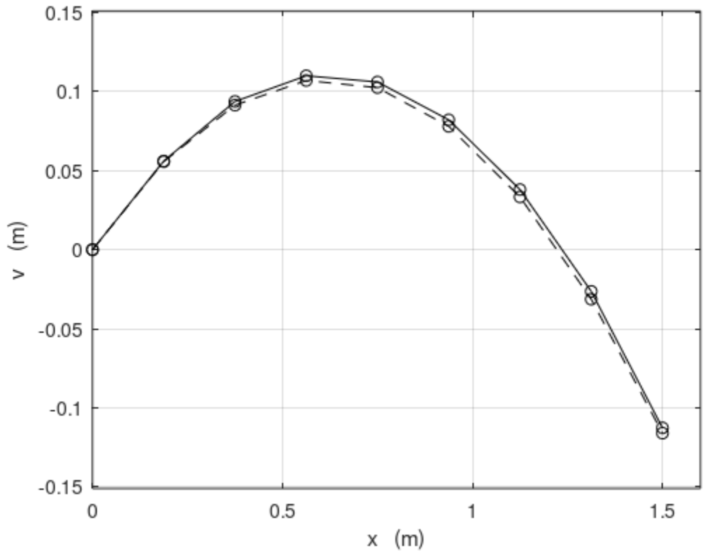

Figure 5. Displacement of beam nodes: - - -, reference; ––, numerical result.

As can be seen in this figure, the proposed nonlinear control technique acting along the beam through piezoelectric actuators is sufficient to bend the flexible structure and maintain the desired shape with good accuracy even under an angular velocity of 10 rad/s.

6. CONCLUSIONS

A set of coupled nonlinear governing equations of motion for a mechanical system consisting of a rotating flexible beam-like structure with a piezoelectric layer bonded to its surface and coupled with a DC motor is derived and analyzed in the closed loop.

The nonlinear control technique named SDRE is used to maintain the angular velocity of the motor axis constant and simultaneously deform the flexible structure into the desired shape.

Even for sufficiently high angular velocities, the proposed nonlinear optimal control technique, according to the results presented here, can achieve the proposed objectives with very good accuracy.

The piezoelectric actuators act along the flexible beam-like structure to deform it according to some prescribed shape and can also maintain this shape when the structure is in rotation.

The idea of controlling angular velocity and deformation of a smart flexible structure at the same time and with the desired accuracy and with the problem being treated as a nonlinear problem without simplifications (linearization), as far as this author is aware, is a nonexistent or very rare result.

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