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*ORIGINAL RESEARCH*

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# QUATTUORTRIGINTIC FUNCTIONAL EQUATION AND ITS HYERS-ULAM STABILITY

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Abstract. This article deals a new Quattuortrigintic functional equation with the general solution and the generalized Hyers-Ulam stability in multi-Banach space and menger probabilistic normed space by employing fixed point technique.

## 1 Introduction and Preliminaries

Stability problem of a functional equation was first posed by Ulam [46] and that was partially answered by Hyers [14] and then generalized by Aoki [2] and Rassias [37] for additive mappings and linear mappings, respectively. In 1994, a generalization of Rassias theorem was obtained by Gâvruta [13] and this idea is known as generalized Hyers-Ulam-Rassias stability. After that, the general stability problems of various functional equations such as additive [23, 24], quadratic [22, 28], cubic [5, 21, 29, 30], quartic [5, 33], quintic and sextic [4, 25, 32], septic and octic [47], nonic [6, 42], decic [3], undecic [40], quattuordecic [41], hexadecic [18], octadecic [26], vigintic [39], viginticduo [17], quattuorvigintic [27, 38, 35] and trigintic [8] and so on.

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The multi-Banach spaces was first investigated by Dales and Polyakov [9]. Theory of multi-Banach spaces is similar to operator sequence space and has some connections with operator spaces and Banach spaces. In 2007, Dales and Moslehian [10] first proved the stability of mappings and also gave some examples on multi-normed spaces. The asymptotic aspects of the quadratic functional equations in multi-normed spaces was investigated by Moslehian, Nikodem and Popa [19] in 2009. In the last two decades, the stability of functional equations on multi-normed spaces was proved by many mathematicians, one can refer [12, 16, 43, 45].

**Definition 1.1.** [10] A multi-norm on $\left\{\wp^k : k \in \mathbb{N}\right\}$ is a sequence $(\|.\|) = (\|.\|_k : k \in \mathbb{N})$ such that $\|.\|_k$ is a norm on $\wp^k$ for each $k \in \mathbb{N}, \|x\|_1 = \|x\|$ for each $x \in \wp$, and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$ :

(1) $\left\|\left(x_{\sigma(1)}, ..., x_{\sigma(k)}\right)\right\|_k = \|(x_1...x_k)\|_k$, for $\sigma \in \Psi_k, x_1, ..., x_k \in \wp$;
(2) $\|(\alpha_1 x_1, ..., \alpha_k x_k)\|_k \leq (\max_{i\in\mathbb{N}_k} |\alpha_i|) \|(x_1...x_k)\|_k$
for $\alpha_1...\alpha_k \in \mathbb{C}, x_1, ..., x_k \in \wp$;
(3) $\|(x_1, ..., x_{k-1}, 0)\|_k = \|(x_1, ..., x_{k-1})\|_{k-1}$, for $x_1, ..., x_{k-1} \in \wp$;
(4) $\|(x_1, ..., x_{k-1}, x_{k-1})\|_k = \|(x_1, ..., x_{k-1})\|_{k-1}$ for $x_1, ..., x_{k-1} \in \wp$.

In this case, we say that $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi-normed space. Suppose that $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi-normed space, and take $k \in \mathbb{N}$. We need the following two properties of multi - norms. They can be found in [10].

$(a)\ \|(x, ...x)\|_k = \|x\|, \forall x \in \wp,$

$$(b) \max_{i\in\mathbb{N}_k} \|x_i\| \leq \|(x_1, ..., x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i\in\mathbb{N}_k} \|x_i\|, \forall x_1, ..., x_k \in \wp.$$

It follows from (b) that if $(\wp, \|.\|)$ is a Banach space, then $(\wp^k, \|.\|_k)$ is a Banach space for each $k \in \mathbb{N}$. In this case, $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi-Banach space.

**Definition 1.2.** A function $\mathfrak{F} : \mathbb{R} \to [0, 1]$ is called a distribution function if it is non- decreasing and left continuous, with sup $\mathfrak{F}(t) = 1$ and inf $\mathfrak{F}(t) = 0$. The class of distribution functions $\mathfrak{F}$ with $\mathfrak{F}(0) = 0$ is denoted by $D_+.\epsilon_0$ is the element of $D_+$

defined by

$$\epsilon_0 = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases}.$$

**Definition 1.3.** A binary operation $\tau : [0,1] \times [0,1] \to [0,1]$ is said to be $t-$ norm if it satisfies the following conditions:
(a) $\tau$ is commutative and associative,
(b) $\tau$ is continuous,
(c) $p\tau 1 = p$ for all $p \in [0,1]$,
(d) $p\tau q \leq r\tau s$ whenever $p \leq r$ and $q \leq s$ for all $p, q, r, s \in [0,1]$.

**Definition 1.4.** Let $\mathfrak{L}$ be a real vector space, $\mathfrak{F}$ a mapping from $\mathfrak{L}$ to $D_+$ (for any $l \in \mathfrak{L}, \mathfrak{F}(l)$ is denoted by $\mathfrak{F}_{\mathfrak{L}}$) and $\tau$ a $t-$ norm. The triple $(\mathfrak{L}, \mathfrak{F}, \tau)$ is called a menger probabilistic normed space, if the following are satisfied:
(a) $\mathfrak{F}_{\mathfrak{L}}(0) = 0$, for all $l \in \mathfrak{L}$,
(b) $\mathfrak{F}_{\mathfrak{L}}(0) = \epsilon_0$ iff $l = \theta$ ($\theta$ is the null vector in $\mathfrak{L}$),
(c) $\mathfrak{F}_{al}(t) = \mathfrak{F}_{\mathfrak{L}}\left(\frac{t}{|a|}\right)$ for all $a \in \mathbb{R}, a \neq 0$ and $l \in \mathfrak{L}$,
(d) $\mathfrak{F}_{l+m}(t_1 + t_2) \geq \mathfrak{F}_l(t_1)\tau\mathfrak{F}_m(t_2)$ for all $l, m \in \mathfrak{L}$ and $t_1, t_2 > 0$.

**Definition 1.5.** Let $X$ be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a generalized metric on $X$ if $d$ satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The following fixed point theorem proved by Diaz and Margolis [11] plays an important role in proving our theorem:

**Theorem 1.6.** *Let $(\mathcal{X}, d)$ be a complete generalized metric space and let $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $\mathcal{L} < 1$. Then for each given element $x \in \mathcal{X}$, either*

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

*for all non-negative integers $n$ or there exists a positive integer $n_0$ such that*
*(i) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty$ for all $n \geq n_0$;*

*(ii) The sequence* $\{\mathcal{J}^n x\}$ *is convergent to a fixed point* $y^*$ *of* $\mathcal{J}$*;*
*(iii)* $y^*$ *is the unique fixed point of* $T$ *in the set* $Y = \{y \in \mathcal{X} : d(\mathcal{J}^{n_0}x, y) < \infty\}$*;*
*(iv)* $d(y, y^*) \leq \frac{1}{1-\mathcal{L}} d(y, \mathcal{J}y)$ *for all* $y \in Y$*.*

Alsina [1], Mihet and Radhu [20] investigated the stability in probabilistic and random normed spaces. In 2009, Shakeri et al.[44] studied the stability of cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x)$$

in Menger probablistic normed spaces. Authors Ravi et al.[31] investigated the stability of cubic functional equation of the form

$$f(2x + y) - 3f(x + y) + f(x - y) = 6f(x) - 3f(y)$$

in menger probabilistic normed spaces.

Motivation from the above developments in the various type of functional equations, in this paper authors introduce a new Quattuortrigintic functional equation satisfied by $g(l) = l^{34}$ of the form

$$\begin{aligned} &g(l + 17m) - 34g(l + 16m) + 561g(l + 15m) - 5984g(l + 14m) \\ &\quad + 46376g(l + 13m) - 278256g(l + 12m) + 1344904g(l + 11m) \\ &\quad - 5379616g(l + 10m) + 18156204g(l + 9m) - 52451256g(l + 8m) \\ &\quad + 131128140g(l + 7m) - 286097760g(l + 6m) + 548354040g(l + 5m) \\ &\quad - 927983760g(l + 4m) + 1391975640g(l + 3m) - 1855967520g(l + 2m) \\ &\quad + 2203961430g(l + m) - 2333606220g(l) + 2203961430g(l - m) \\ &\quad - 1855967520g(l - 2m) + 1391975640g(l - 3m) - 927983760g(l - 4m) \\ &\quad + 548354040g(l - 5m) - 286097760g(l - 6m) + 131128140g(l - 7m) \\ &\quad - 52451256g(l - 8m) + 18156204g(l - 9m) - 5379616g(l - 10m) \\ &\quad + 1344904g(l - 11m) - 278256g(l - 12m) + 46376g(l - 13m) \\ &\quad - 5984g(l - 14m) + 561g(l - 15m) - 34g(l - 16m) + g(l - 17m) = 34!g(m) \end{aligned} \tag{1.1}$$

for all $l, m \in \mathfrak{L}$, where $34! = 2.95232799 \times 10^{38}$. Also, author obtain its general solution and the generalized Hyers-Ulam stability in multi-Banach space and menger probabilistic normed space by employing fixed point technique.

Let $\mathfrak{L}$ and $\mathfrak{M}$ be real vector spaces. For convenience, we use the abbreviation for a mapping $g : \mathfrak{L} \to \mathfrak{M}$ as $\mathcal{D}g(l, m) = 0$, where $\mathcal{D}g(l, m)$ represents (1.1).

## 2 General soultion of Quattuortrigintic functional equation

**Theorem 2.1.** *Let $\mathfrak{L}$ and $\mathfrak{M}$ be vector spaces. If $g : \mathfrak{L} \to \mathfrak{M}$ is the function (1.1) for all $l, m \in \mathfrak{L}$, then $g$ is a Quattuortrigintic mapping.*

*Proof.* Substituting $l = 0$ and $m = 0$ in (1.1), we obtain that $g(0) = 0$. Substituting $(l, m)$ with $(l, l)$ and $(l, -l)$ in (1.1), respectively, and subtracting two resulting equations, we can arrive at $g(-l) = g(l)$, that is to say, $g$ is an even function. Letting $(l, m)$ by $(17l, l)$ and $(0, 2l)$ respectively in (1.1), and subtracting the two resulting equations, we arrive at

$$\begin{aligned}
&34g(33l) - 595g(32l)5984g(31l) - 45815g(30l) + 278256g(29l) \\
&\quad - 1350888g(28l) + 5379616g(27l) - 18109828g(26l) + 52451256g(25l) \\
&\quad - 131406396g(24l) + 286097760g(23l) - 547009136g(22l) + 927983760g(21l) \\
&\quad - 1397355256g(20l) + 1855967520g(19l) - 2185805226g(18l) + 2333606220g(17l) \\
&\quad - 2256412686g(16l) + 1855967520g(15l) - 1260847500g(14l) + 927983760g(13l) \\
&\quad - 834451800g(12l) + 286097760g(11l) + 417225900g(10l) + 52451256g(9l) \\
&\quad - 946139964g(8l) + 5379616g(7l) + 1390630736g(6l) + 278256g(5l) \\
&\quad - 1856013896g(4l) + 5984g(3l) - 1.476163995 \times 10^{38}g(2l) + 34!g(l) = 0
\end{aligned} \tag{2.1}$$

for all $l \in \mathfrak{L}$. Substituting $l = 16l$ and $m = l$ in (1.1), further multiplying the resulting equation by 34, and subtracting the obtained result from (2.1), we get

$561g(32l) - 13090g(31l) + 157641g(30l) - 1298528g(29l) + 8109816g(28l)$
$-4037120g(27l) + 164797116g(26l) - 564859680g(25l) + 1651936308g(24l)$
$-4172259000g(23l) + 9180314704g(22l) - 1.77160536 \times 10^{10}g(21l)$
$+3.015409258 \times 10^{10}g(20l) - 4.547120424 \times 10^{10}g(19l) + 6.091709045 \times 10^{10}g(18l)$

$-7.26010824 \times 10^{10} g(17l) + 7.708619879 \times 10^{10} g(16l) - 7.30787211 \times 10^{10} g(15l)$
$+6.184204818 \times 10^{10} g(14l) - 4.6399188 \times 10^{10} g(13l) + 3.071699604 \times 10^{10} g(12l)$
$-1.83579396 \times 10^{10} g(11l) + 1.014454974 \times 10^{10} g(10l) - 4405905504 g(9l)$
$+837202740 g(8l) - 611931320 g(7l) + 1537537680 g(6l) - 45448480 g(5l)$

$$- 1846553192 g(4l) - 1570800 g(3l) - \frac{34!}{2} g(2l) + 34!(35) g(l) = 0 \qquad (2.2)$$

for all $l \in \mathfrak{L}$. Letting $l = 15l$ and $m = l$ in (1.1), further multiplying the resulting equation by 561, and subtracting the obtained result from (2.2), we arrive
$5984 g(31l) - 157080 g(30l) + 2058496 g(29l) - 17907120 g(28l) + 115754496 g(27l)$
$-589694028 g(26l) + 2453104896 g(25l) - 8533694136 g(24l) + 2.525289562 \times 10^{10} g(23l)$
$-6.438257184 \times 10^{10} g(22l) + 1.427847898 \times 10^{11} g(21l) - 2.774725239 \times 10^{11} g(20l)$
$+4.751276851 \times 10^{11} g(19l) - 7.199812436 \times 10^{11} g(18l) + 9.685966966 \times 10^{11} g(17l)$
$-1.159336163 \times 10^{12} g(16l) + 1.236074368 \times 10^{12} g(15l) - 1.174580314 \times 10^{12} g(14l)$
$+9.947985907 \times 10^{11} g(13l) - 7.50181338 \times 10^{11} g(12l) + 5.022409498 \times 10^{11} g(11l)$
$-2.974820667 \times 10^{11} g(10l) + 1.560949379 \times 10^{11} g(9l) - 7.27256838 \times 10^{10} g(8l)$
$+2.88132233 \times 10^{10} g(7l) - 8612092764 g(6l) + 2972516096 g(5l) - 2601044336 g(4l)$

$$+ 154530816 g(3l) - \frac{34!}{2} g(2l) + 34!(596) g(l) = 0 \qquad (2.3)$$

for all $l \in \mathfrak{L}$. Taking $l = 14l$ and replacing $m = l$ in (1.1), further multiplying the resulting equation by 5984, and subtracting the obtained result from (2.3), we have
$46376 g(30l) - 1298528 g(29l) + 17901136 g(28l) - 161759488 g(27l) + 1075389876 g(26l)$
$-5594800640 g(25l) + 2.365792801 \times 10^{10} g(24l) - 8.339382912 \times 10^{10} g(23l)$
$+2.494857441 \times 10^{11} g(22l) - 6.41886 \times 10^{11} g(21l) + 1.434536472 \times 10^{12} g(20l)$
$-2.80622289 \times 10^{12} g(19l) + 4.833073576 \times 10^{12} g(18l) - 7.360985533 \times 10^{12} g(17l)$
$+9.946773477 \times 10^{12} g(16l) - 1.195243083 \times 10^{13} g(15l) + 1.278971931 \times 10^{13} g(14l)$
$-1.219370661 \times 10^{13} g(13l) + 1.03559283 \times 10^{12} g(12l) - 7.82734128 \times 10^{12} g(11l)$
$+5.255572753 \times 10^{12} g(10l) - 3.125255638 \times 10^{12} g(9l) + 1.639283312 \times 10^{12} g(8l)$
$-7.558575665 \times 10^{11} g(7l) + 3.052562231 \times 10^{11} g(6l) - 1.056742086 \times 10^{11} g(5l)$

$$+ 2.959057781 \times 10^{10} g(4l) - 7893380704 g(3l) - \frac{34!}{2} g(2l) + 34!(6580) g(l) = 0 \quad (2.4)$$

for all $l \in \mathfrak{L}$. Taking $l = 13l$ and replacing $m = l$ in (1.1), further multiplying the resulting equation by 46376, and subtracting the obtained result from (2.4), we arrive at

$278256g(29l) - 8115800g(28l) + 115754496g(27l) - 1075343500g(26l)$
$+7309599616g(25l) - 3.87133399 \times 10^{10}g(24l) + 1.660912425 \times 10^{11}g(23l)$
$-5.925263726 \times 10^{11}g(22l) + 1.790593448 \times 10^{12}g(21l) - 4.646662149 \times 10^{12}g(20l)$
$+1.046184683 \times 10^{13}g(19l) - 2.059739338 \times 10^{13}g(18l) + 3.567518932 \times 10^{13}g(17l)$
$-5.46074888 \times 10^{13}g(16l) + 7.411991888 \times 10^{13}g(15l) - 8.942119597 \times 10^{13}g(14l)$
$+9.602961545 \times 10^{13}g(13l) - 9.185498698 \times 10^{13}g(12l) + 7.824500843 \times 10^{13}g(11l)$
$-5.929868953 \times 10^{13}g(10l) + 3.991091922 \times 10^{13}g(9l) - 2.379118365 \times 10^{13}g(8l)$
$+1.251221215 \times 10^{13}g(7l) - 5.775942398 \times 10^{12}g(6l) + 2.32680524 \times 10^{12}g(5l)$
$-8.124215853 \times 10^{11}g(4l) + 2.415932677 \times 10^{11}g(3l)$

$$- -1.476163995 \times 10^{38}g(2l) + 34!(54956)g(l) = 0 \tag{2.5}$$

for all $l \in \mathfrak{L}$. Letting $l = 12l$ and $m = l$ in (1.1), further multiplying the resulting equation by 278256, and subtracting the obtained result from (2.5), we have

$1344904g(28l) - 40347120g(27l) + 589740404g(26l) - 5594800640g(25l)$
$+3.871306164 \times 10^{10}g(24l) - 2.081363649 \times 10^{11}g(23l) + 9.043840571 \times 10^{11}g(22l)$
$-3.261479252 \times 10^{12}g(21l) + 9.948214541 \times 10^{12}g(20l) - 2.60253449 \times 10^{13}g(19l)$
$+5.901102492 \times 10^{13}g(18l) - 1.169076125 \times 10^{14}g(17l) + 2.036095603 \times 10^{14}g(16l)$
$-3.132056548 \times 10^{14}g(15l) + 4.270129023 \times 10^{14}g(14l) - 5.172358762 \times 10^{14}g(13l)$
$+5.574849454 \times 10^{14}g(12l) - 5.350204832 \times 10^{14}g(11l) + 4.571354087 \times 10^{14}g(10l)$
$-3.474146545 \times 10^{14}g(9l) + 2.344258655 \times 10^{14}g(8l) - 1.400705896 \times 10^{14}g(7l)$
$+7.383247591 \times 10^{13}g(6l) - 3.416038676 \times 10^{12}g(5l) + 1.378246456 \times 10^{13}g(4l)$

$$- 4.810635534 \times 10^{12}g(3l) - \frac{34!}{2}g(2l) + 34!(331212)g(l) = 0 \tag{2.6}$$

for all $l \in \mathfrak{L}$. Setting $l = 11l$ and $m = l$ in (1.1), further multiplying the resulting equation by 1344904, and subtracting the obtained result from (2.6), we get

$5379616g(27l) - 164750740g(26l) + 2453104896g(25l) - 2.365820626 \times 10^{10}g(24l)$
$+1.660912425 \times 10^{11}g(23l) - 9.043827122 \times 10^{11}g(22l) + 3.973587825 \times 10^{12}g(21l)$
$-1.447013684 \times 10^{13}g(20l) + 4.45165591 \times 10^{13}g(19l) - 1.173437351 \times 10^{14}g(18l)$
$+2.678664093 \times 10^{14}g(17l) - 5.338739815 \times 10^{14}g(16l) + 9.34843416 \times 10^{14}g(15l)$
$-1.445060704 \times 10^{15}g(14l) + 1.978862265 \times 10^{15}g(13l) - 2.406631598 \times 10^{15}g(12l)$
$+2.603455856 \times 10^{15}g(11l) - 2.506981134 \times 10^{15}g(10l) + 2.148683487 \times 10^{15}g(9l)$
$-1.637647741 \times 10^{15}g(8l) + 1.107978481 \times 10^{15}g(7l) - 6.636510672 \times 10^{14}g(6l)$

$$+3.506136808\times10^{14}g(5l)-1.625730499\times10^{14}g(4l)+6.573931637\times10^{13}g(3l)$$
$$--1.476163995\times10^{38}g(2l)+34!(1676116)g(l)=0 \tag{2.7}$$

for all $l\in\mathfrak{L}$. Taking $l=10l$ and replacing $m=l$ in (1.1), further multiplying the resulting equation by 5379616, and subtracting the obtained result from (2.7), we arrive at

$$18156204g(26l)-564859680g(25l)+8533415880g(24l)-8.339382912\times10^{10}g(23l)$$
$$+5.925277175\times10^{11}g(22l)-3.261479252\times10^{12}g(21l)+1.447013146\times10^{13}g(20l)$$
$$-5.315684643\times10^{13}g(19l)+1.648238809\times10^{14}g(18l)-4.375526307\times10^{14}g(17l)$$
$$+1.005222106\times10^{15}g(16l)-2.015090751\times10^{15}g(15l)+3.547135579\times10^{15}g(14l)$$
$$-5.509432159\times10^{15}g(13l)+7.577760968\times10^{15}g(12l)-9.253010316\times10^{15}g(11l)$$
$$+1.004692422\times10^{16}g(10l)-9.707782685\times10^{15}g(9l)+8.346744825\times10^{15}g(8l)$$
$$-6.380315949\times10^{15}g(7l)+4.328545399\times10^{15}g(6l)-2.599323504\times10^{15}g(5l)$$
$$+1.376555229\times10^{15}g(4l)-6.399292087\times10^{14}g(3l)$$
$$--1.476163995\times10^{38}g(2l)+34!(7055732)g(l)=0 \tag{2.8}$$

for all $l\in\mathfrak{L}$. Letting $l=9l$ and $m=l$ in (1.1),further multiplying the resulting equation 18156204, and subtracting the obtained result from (2.8), we arrive at

$$52451256g(25l)-1652214564g(24l)+2.525289562\times10^{10}g(23l)$$
$$-2.494843992\times10^{11}g(22l)+1.790593448\times10^{12}g(21l)-9.948219921\times10^{12}g(20l)$$
$$+4.45165591\times10^{13}g(19l)-1.648238628\times10^{14}g(18l)+5.147630733\times10^{14}g(17l)$$
$$-1.375567154\times10^{15}g(16l)+3.179358543\times10^{15}g(15l)-6.408892235\times10^{15}g(14l)$$
$$+1.3392303\times10^{16}g(13l)-1.769523271\times10^{16}g(12l)+2.444431459\times10^{16}g(11l)$$
$$-2.996864911\times10^{16}g(10l)+3.26616479\times10^{16}g(9l)-3.166882852\times10^{16}g(8l)$$
$$+2.731700958\times10^{16}g(7l)-2.094445847\times10^{16}g(6l)+1.42494476\times10^{16}g(5l)$$
$$-8.580314598\times10^{15}g(4l)+4.559572159\times10^{15}g(3l)-\frac{34!}{2}g(2l)+34!(25211936)g(l)=0 \tag{2.9}$$

for all $l\in\mathfrak{L}$. Setting $l=8l$ and $m=l$ in (1.1), further multiplying the resulting equation by 52451256, and subtracting the obtained result from (2.9), we arrive at

$$131128140g(24l)-4172259000g(23l)+6.438391676\times10^{10}g(22l)-6.41886\times10^{11}g(21l)$$
$$+4.646656769\times10^{12}g(20l)-2.60253449\times10^{13}g(19l)+1.173437532\times10^{14}g(18l)$$
$$-4.375526307\times10^{14}g(17l)+1.375567102\times10^{15}g(16l)-3.698477097\times10^{15}g(15l)$$
$$+8.597294616\times10^{15}g(14l)-1.742262783\times10^{16}g(13l)+3.097868105\times10^{16}g(12l)$$

$-9.073175051 \times 10^{16} g(11l) + 6.737917841 \times 10^{16} g(10l) - 8.293889733 \times 10^{16} g(9l)$
$+9.073175051 \times 10^{16} g(8l) - 8.828356503 \times 10^{16} g(7l) + 7.640368292 \times 10^{16} g(6l)$
$-5.876385552 \times 10^{16} g(5l) + 4.010819404 \times 10^{16} g(4l) - 2.427282788 \times 10^{16} g(3l)$

$$- -1.476163995 \times 10^{38} g(2l) + 34!(77663192) g(l) = 0 \tag{2.10}$$

for all $l \in \mathfrak{L}$. Replacing $l = 7l$ and $m = l$ in (1.1), further multiplying the resulting equation by 131128140, and subtracting the obtained result from (2.10), we have
$286097760 g(23l) - 9178969796 g(22l) + 1.427847898 \times 10^{11} g(21l)$
$-1.434541852 \times 10^{12} g(20l) + 1.046184683 \times 10^{13} g(19l) - 5.901100677 \times 10^{13} g(18l)$
$+2.678664093 \times 10^{14} g(17l) - 1.005222158 \times 10^{15} g(16l) + 3.179358543 \times 10^{15} g(15l)$
$-8.597294484 \times 10^{15} g(14l) + 2.009283929 \times 10^{16} g(13l) - 4.092596428 \times 10^{16} g(12l)$
$+7.311822835 \times 10^{16} g(11l) - 1.151479983 \times 10^{17} g(10l) + 1.604306759 \times 10^{17} g(9l)$
$-1.98269686 \times 10^{17} g(8l) + 2.177186628 \times 10^{17} g(7l) - 2.126037612 \times 10^{17} g(6l)$
$+1.846422005 \times 10^{17} g(5l) - 1.425953373 \times 10^{17} g(4l) + 9.811737556 \times 10^{16} g(3l)$

$$-\frac{34!}{2} g(2l) + 34!(208791332) g(l) = 0 \tag{2.11}$$

for all $l \in \mathfrak{L}$. Letting $l = 6l$ and $m = l$ in (1.1), further multiplying the resulting equation by 286097760, and subtracting the obtained result from (2.11), we have
$548354040 g(22l) - 1.77160536 \times 10^{10} g(21l) + 2.774671442 \times 10^{11} g(20l)$
$-2.80622289 \times 10^{12} g(19l) + 2.059741154 \times 10^{13} g(18l) - 1.169076125 \times 10^{14} g(17l)$
$+5.338739293 \times 10^{14} g(16l) - 2.015090751 \times 10^{15} g(15l) + 6.408892366 \times 10^{15} g(14l)$
$-1.742262783 \times 10^{16} g(13l) + 4.0925964 \times 10^{16} g(12l) - 8.376463446 \times 10^{16} g(11l)$
$+1.503460865 \times 10^{17} g(10l) - 2.378105972 \times 10^{17} g(9l) + 3.327201761 \times 10^{17} g(8l)$
$-4.128430336 \times 10^{17} g(7l) + 4.551153595 \times 10^{17} g(6l) - 4.462910018 \times 10^{17} g(5l)$
$+3.899319089 \times 10^{17} g(4l) - 3.053181863 \times 10^{17} g(3l)$

$$- -1.476163995 \times 10^{38} g(2l) + 34!(494889092) g(l) = 0 \tag{2.12}$$

for all $l \in \mathfrak{L}$. Setting $l = 5l$ and $m = l$ in (1.1), further multiplying the resulting equation by 548354040, and subtracting the obtained result from (2.12), we have
$927983760 g(21l) - 3.015947224 \times 10^{10} g(20l) + 4.751276852 \times 10^{11} g(19l)$
$-4.833055422 \times 10^{12} g(18l) + 3.567518932 \times 10^{13} g(17l) - 2.036096128 \times 10^{14} g(16l)$
$+9.348434163 \times 10^{14} g(15l) - 3.547135448 \times 10^{15} g(14l) + 1.13392303 \times 10^{16} g(13l)$
$-3.097868188 \times 10^{16} g(12l) + 9.311824671 \times 10^{16} g(11l) - 1.503463743 \times 10^{17} g(10l)$

$+2.71056328 \times 10^{17} g(9l) - 4.306007201 \times 10^{17} g(8l) + 6.050368369 \times 10^{17} g(7l)$
$-7.541732782 \times 10^{17} g(6l) + 8.363013309 \times 10^{17} g(5l) - 8.285752731 \times 10^{17} g(4l)$

$$+ 7.411709595 \times 10^{17} g(3l) - \frac{34!}{2} g(2l) + 34!(1043243132) g(l) = 0 \tag{2.13}$$

for all $l \in \mathfrak{L}$. Letting $l = 4l$ and $m = l$ in (1.1), further multiplying the resulting equation by 927983760, and subtracting the obtained result from (2.13), we get
$1391975640 g(20l) - 4.547120416 \times 10^{10} g(19l) + 7.199993974 \times 10^{11} g(18l)$
$-7.360985533 \times 10^{12} g(17l) + 5.460743657 \times 10^{13} g(16l) - 3.132056548 \times 10^{14} g(15l)$
$+1.445060835 \times 10^{15} g(14l) - 5.509433087 \times 10^{15} g(13l) + 1.769526343 \times 10^{16} g(12l)$
$-4.856705829 \times 10^{16} g(11l) + 1.151532538 \times 10^{17} g(10l) - 2.37850352 \times 10^{17} g(9l)$
$+4.308113557 \times 10^{17} g(8l) - 6.879420004 \times 10^{17} g(7l) + 9.731266357 \times 10^{17} g(6l)$
$-1.225787746 \times 10^{18} g(5l) + 1.385647315 \times 10^{18} g(4l) - 1.42575424 \times 10^{18} g(3l)$

$$- -1.476163995 \times 10^{38} g(2l) + 34!(1971226892) g(l) = 0 \tag{2.14}$$

for all $l \in \mathfrak{L}$. Taking $l = 3l$ and replacing $m = l$ in (1.1), further multiplying the resulting equation by 1391975640, and subtracting the obtained result from (2.14), we arrive
$1855967520 g(19l) - 6.08989366 \times 10^{10} g(18l) + 9.68596697 \times 10^{11} g(17l)$
$-9.94682571 \times 10^{12} g(16l) + 7.4119919187 \times 10^{13} g(15l) - 4.270141631 \times 10^{14} g(14l)$
$+1.978908665 \times 10^{15} g(13l) - 7.57851115 \times 10^{15} g(12l) + 2.44521419 \times 10^{16} g(11l)$
$-6.74384771 \times 10^{16} g(10l) + 1.607780862 \times 10^{17} g(9l) - 3.343561836 \times 10^{17} g(8l)$
$+6.112770823 \times 10^{17} g(7l) - 9.897425403 \times 10^{17} g(6l) + 1.430684701 \times 10^{118} g(5l)$
$-1.864740484 \times 10^{18} g(4l) + 2.220809885 \times 10^{18} g(3l)$

$$- \frac{34!}{2} g(2l) + 34!(3363202532) g(l) = 0 \tag{2.15}$$

for all $l \in \mathfrak{L}$. Taking $l = 2l$ and replacing $m = l$ in (1.1), further multiplying the resulting equation by 1855967520, and subtracting the obtained result from (2.15), we arrive
$2203961430 g(18l) - 7.2601082 \times 10^{10} g(17l) + 1.15928393 \times 10^{12} g(16l)$
$-1.19542868 \times 10^{13} g(15l) + 8.948303802 \times 10^{13} g(14l) - 5.182306748 \times 10^{14} g(13l)$
$+2.416987526 \times 10^{15} g(12l) - 9.331255324 \times 10^{15} g(11l) + 3.042578451 \times 10^{16} g(10l)$
$-8.508758077 \times 10^{16} g(9l) + 2.06616359 \times 10^{17} g(8l) - 4.401475303 \times 10^{17} g(7l)$

$$+8.299130048\times10^{17}g(6l)-1.396146444\times10^{18}g(5l)+2.110863102\times10^{18}g(4l)$$

$$-2.887398233\times10^{18}g(3l)-\frac{34!}{2}g(2l)+34!(5219170052)g(l)=0 \tag{2.16}$$

for all $l\in\mathfrak{L}$. Taking $l=l$ and replacing $m=l$ in (1.1), further multiplying the resulting equation by 2203961430, and subtracting the obtained result from (2.16), we arrive

$$2333606220g(17l)-7.934261148\times10^{10}g(16l)+1.309153089\times10^{12}g(15l)$$
$$-1.396429962\times10^{13}g(14l)+1.082233221\times10^{14}g(13l)-6.493399323\times10^{14}g(12l)$$
$$+3.13847634\times10^{15}g(11l)-1.255390536\times10^{16}g(10l)+4.236943059\times10^{14}g(9l)$$
$$-1.224005772\times10^{17}g(8l)+3.060014431\times10^{17}g(7l)-6.676395123\times10^{17}g(6l)$$
$$+1.279642399\times10^{18}g(5l)-2.165548674\times10^{18}g(4l)+3.248323012\times10^{18}g(3l)$$

$$--1.476163995\times10^{38}g(2l)+34!(8589934592)g(l)=0 \tag{2.17}$$

for all $l\in\mathfrak{L}$. Taking $l=0$ and replacing $m=l$ in (1.1), further multiplying the resulting equation by 2333606220, and subtracting the obtained result from (2.17), we arrive

$$g(2l)=2^{34}g(l) \tag{2.18}$$

where $2^{34}=1.717986918\times10^{10}$ for all $l\in\mathfrak{L}$. □

## 3 Hyers-Ulam Stability of Functional Equation (1.1) in Multi-Banach Spaces

**Theorem 3.1.** *Let $\mathfrak{L}$ be an linear space and let $\left((\mathfrak{M}^k,\|.\|_k):k\in\mathbb{N}\right)$ be a multi-Banach space. Suppose that $\eta$ is a non-negative real number and $g:\mathfrak{L}\to\mathfrak{M}$ be a function fulfills*

$$\sup_{k\in\mathbb{N}}\|(\mathcal{D}g(l_1,m_1),...,\mathcal{D}g(l_k,m_k))\|_k\le\delta \tag{3.1}$$

$\forall l_1,...,l_k,m_1,...,m_k\in\mathfrak{L}$. *Then there exists a unique Quattuortrigintic mapping $\mathfrak{T}:\mathfrak{L}\to\mathfrak{M}$ such that*

$$\sup_{k\in\mathbb{N}}\|(g(l_1)-\mathfrak{T}(l_1),...,g(l_k)-\mathfrak{T}(l_k))\|_k\le\frac{1.71986919\times10^{10}}{34!(1.71986917\times10^{10})}\eta. \tag{3.2}$$

*for all $l_i\in\mathfrak{L}$, where $i=1,2,...,k$.*

*Proof.* Letting $(l_i, m_i)$ by $(17l_i, l_i)$ and $(0, 2l_i)$ respectively in (3.1), and subtracting the two resulting equations, we arrive at

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(34g(33l_1) - 595g(32l_1) + 5984g(31l_1) - 45815g(30l_1) + 278256g(29l_1)\\
&-1350888g(28l_1) + 5379616g(27l_1) - 18109828g(26l_1) + 52451256g(25l_1)\\
&-131406396g(24l_1) + 286097760g(23l_1) - 547009136g(22l_1) + 927983760g(21l_1)\\
&-1397355256g(20l_1) + 1855967520g(19l_1) - 2185805226g(18l_1)\\
&+2333606220g(17l_1) - 2256412686g(16l_1) + 1855967520g(15l_1) - 1260847500g(14l_1)\\
&+927983760g(13l_1) - 834451800g(12l_1) + 286097760g(11l_1) + 417225900g(10l_1)\\
&+52451256g(9l_1) - 946139964g(8l_1) + 5379616g(7l_1) + 1390630736g(6l_1)\\
&+278256g(5l_1) - 1856013896g(4l_1) + 5984g(3l_1) - \frac{34!}{2}g(2l_1) + 34!g(l_1), ..., 34g(33l_k)\\
&-595g(32l_k) + 5984g(31l_k) - 45815g(30l_k) + 278256g(29l_k) - 1350888g(28l_k)\\
&+5379616g(27l_k) - 18109828g(26l_k) + 52451256g(25l_k) - 131406396g(24l_k)\\
&+286097760g(23l_k) - 547009136g(22l_k) + 927983760g(21l_k) - 1397355256g(20l_k)\\
&+1855967520g(19l_k) - 2185805226g(18l_k) + 2333606220g(17l_k) - 2256412686g(16l_k)\\
&+1855967520g(15l_k) - 1260847500g(14l_k) + 927983760g(13l_k) - 834451800g(12l_k)\\
&+286097760g(11l_k) + 417225900g(10l_k) + 52451256g(9l_k) - 946139964g(8l_k)\\
&+5379616g(7l_k) + 1390630736g(6l_k) + 278256g(5l_k) - 1856013896g(4l_k)\\
&\qquad +5984g(3l_k) - \frac{34!}{2}g(2l_k) + 34!g(l_k)\Big)\Big\|_k \leq \frac{3}{2}\eta \qquad (3.3)
\end{aligned}$$

for all $l_i \in \mathfrak{L}$, where $i = 1, 2, .., k$. Taking $l_i = 16l_i$ and replacing $m_i = l_i$ in (3.1), further multiplying the resulting equation by 32, and subtracting the obtained result from (3.3), we arrive at

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(561g(32l_1) - 13090g(31l_1) + 157641g(30l_1) - 1298528g(29l_1)\\
&+8109816g(28l_1) - 4037120g(27l_1) + 164797116g(26l_1) - 564859680g(25l_1)\\
&+1651936308g(24l_1) - 4172259000g(23l_1) + 9180314704g(22l_1) - 1.77160536 \times 10^{10}g(21l_1)\\
&+3.015409258 \times 10^{10}g(20l_1) - 4.547120424 \times 10^{10}g(19l_1) + 6.091709045 \times 10^{10}g(18l_1)\\
&-7.26010824 \times 10^{10}g(17l_1) + 7.708619879 \times 10^{10}g(16l_1) - 7.30787211 \times 10^{10}g(15l_1)\\
&+6.184204818 \times 10^{10}g(14l_1) - 4.6399188 \times 10^{10}g(13l_1) + 3.071699604 \times 10^{10}g(12l_1)\\
&-1.83579396 \times 10^{10}g(11l_1) + 1.014454974 \times 10^{10}g(10l_1) - 4405905504g(9l_1)\\
&+837202740g(8l_1) - 611931320g(7l_1) + 1537537680g(6l_1) - 45448480g(5l_1)\\
&-1846553192g(4l_1) - 1570800g(3l_1) - \frac{34!}{2}g(2l_1) + 34!(33)g(l_1), ..., 561g(32l_k)\\
&-13090g(31l_k) + 157641g(30l_k) - 1298528g(29l_k) + 8109816g(28l_k) - 4037120g(27l_k)
\end{aligned}$$

$$+164797116g(26l_k) - 564859680g(25l_k) + 1651936308g(24l_k) - 4172259000g(23l_k)$$
$$+9180314704g(22l_k) - 1.77160536 \times 10^{10}g(21l_k) + 3.015409258 \times 10^{10}g(20l_k)$$
$$-4.547120424 \times 10^{10}g(19l_k) + 6.091709045 \times 10^{10}g(18l_k) - 7.26010824 \times 10^{10}g(17l_k)$$
$$+7.708619879 \times 10^{10}g(16l_k) - 7.30787211 \times 10^{10}g(15l_k) + 6.184204818 \times 10^{10}g(14l_k)$$
$$-4.6399188 \times 10^{10}g(13l_k) + 3.071699604 \times 10^{10}g(12l_k) - 1.83579396 \times 10^{10}g(11l_k)$$
$$+1.014454974 \times 10^{10}g(10l_k) - 4405905504g(9l_k) + 837202740g(8l_k)$$
$$-611931320g(7l_k) + 1537537680g(6l_k) - 45448480g(5l_k) - 1846553192g(4l_k)$$

$$-1570800g(3l_k) - \frac{34!}{2}g(2l_k) + 34!(35)g(l_k)\Big)\Big\|_k \leq \frac{71}{2}\eta \tag{3.4}$$

for all $l_i \in \mathfrak{L}$, where $i = 1, 2, .., k$. Applying the same procedure of Theorem 2.1 and using (2.18), we get

$$\sup_{k\in\mathbb{N}} \left\| \left( g(l_1) - \frac{1}{1.71986918 \times 10^{10}} g(2l_1), ..., g(l_k) - \frac{1}{1.71986918 \times 10^{10}} g(2l_k) \right) \right\|_k$$
$$\leq \frac{1.71986919 \times 10^{10}}{(34!)(1.71986918 \times 10^{10})}\eta \tag{3.5}$$

for all $l_i \in \mathfrak{L}$, where $i = 1, 2, .., k$.
Let $\Delta = \{h : \mathfrak{L} \to \mathfrak{M} | h(0) = 0\}$ and introduce the generalized metric $d$ defined on $\Delta$ by

$$d(p, q) = \inf \left\{ \lambda \in [0, \infty] | \sup_{k\in\mathbb{N}} \|(p(l_1) - q(l_1), ..., p(l_k) - q(l_k))\|_k \leq \lambda \quad \forall \quad l_1, ..., l_k \in \mathfrak{L} \right\}.$$

Then it is easy to show that $d$ is a complete generalized metric space on $\Delta$. See [20].
We define an operator $\mathfrak{J} : \Delta \to \Delta$ by

$$\mathfrak{J}p(l) = \frac{1}{1.71986918 \times 10^{10}} p(2l) \ \forall l \in \mathfrak{L}.$$

We assert that $\mathfrak{J}$ is a strictly contractive operator. Given $p, q \in \Delta$, let $\lambda \in (0, \infty)$ be an arbitary constant with $d(p, q) \leq \lambda$. By the definition of $d$, it follows that

$$\sup_{k\in\mathbb{N}} \|(p(l_1) - q(l_1), ..., p(l_k) - q(l_k))\|_k \leq \lambda,$$

for all $l_1, ..., l_k \in \mathfrak{L}$. Therefore,

$$\sup_{k\in\mathbb{N}} \|(\mathfrak{J}p(l_1) - \mathfrak{J}q(l_1), ..., \mathfrak{J}p(l_k) - \mathfrak{J}q(l_k))\|_k \leq \sup_{k\in\mathbb{N}} \left\| \left( \frac{1}{1.71986918 \times 10^{10}} p(2l_1) \right.\right.$$
$$\left.\left. - \frac{1}{1.71986918 \times 10^{10}} q(2l_1), ..., \frac{1}{1.71986918 \times 10^{10}} p(2l_k) - \frac{1}{1.71986918 \times 10^{10}} q(2l_k) \right) \right\|_k$$
$$\leq \frac{1}{1.71986918 \times 10^{10}} \lambda$$

for all $l_1, ..., l_k \in \mathfrak{L}$. Hence, it holds that $d(\mathfrak{J}p, \mathfrak{J}q) \leq \frac{1}{1.71986918 \times 10^{10}} \lambda$.
*i.e.*, $d(\mathfrak{J}p, \mathfrak{J}q) \leq \frac{1}{1.71986918 \times 10^{10}} d(p, q)$
$\forall p, q \in \Delta$. This means that $\mathcal{J}$ is strictly contractive operator on $\Delta$ with the Lipschitz constant $\mathcal{L} = \frac{1}{2^{34}}$.
By (3.5), we have $d(\mathfrak{J}u, u) \leq \frac{1.71986919 \times 10^{10}}{(34!)(1.71986918 \times 10^{10})} \eta$. According to Theorem 1.6, we deduce the existence of a fixed point of $\mathfrak{J}$ that is the existence of mapping $\mathfrak{T} : \mathfrak{L} \to \mathfrak{M}$ such that

$$\mathfrak{T}(2l) = 2^{34} \mathfrak{T}(l)$$

for all $l \in \mathfrak{L}$. Moreover, we have $d(\mathfrak{J}^n u, \mathfrak{T}) \to 0$, which implies

$$\mathfrak{T}(l) = \lim_{n\to\infty} \mathfrak{J}^n u(l) = \lim_{n\to\infty} \frac{u(2^n l)}{2^{34n}}$$

for all $l \in \mathfrak{L}$. Also, $d(u, \mathfrak{T}) \leq \frac{1}{1 - \mathcal{L}} d(\mathfrak{J}u, u)$ implies the inequality

$$d(u, \mathfrak{T}) \leq \frac{1}{1 - \frac{1}{2^{34}}} d(\mathfrak{J}u, u)$$
$$\leq \frac{1.71986919 \times 10^{10}}{(34!)(1.71986917 \times 10^{10})} \eta.$$

Setting $l_1 = ... = l_k = 2^n l, m_1 =, ..., = m_k = 2^n m$ in (3.1) and divide both sides by $2^{34n}$. Then, using property (a) of multi-norms, we obtain

$$\|\mathcal{D}\mathfrak{T}(l, m)\| = \lim_{n\to\infty} \frac{1}{2^{34n}} \|Du(2^n l, 2^n m)\| = 0$$

for all $l, m \in \mathfrak{L}$. Hence $\mathfrak{T}$ is Quattuortrigintic mapping.
The uniqueness of $\mathfrak{T}$ follows from the fact that $\mathfrak{T}$ is the unique fixed point of $\mathfrak{J}$ with

the property that there exists $\ell \in (0,\infty)$ such that

$$\sup_{k\in\mathbb{N}} \|(g(l_1) - \mathfrak{T}(l_1), ..., g(x_k) - \mathfrak{T}(l_k))\|_k \leq \ell$$

for all $l_1, ..., l_k \in \mathfrak{L}$. This completes the proof of the Theorem. □

**Corollary 3.2.** *Let $\mathfrak{L}$ be a linear space, and let $(\mathfrak{M}^k, \|.\|_k)$ be a multi-Banach space. Let $\alpha > 0, 0 < a < 34$ and $g : \mathfrak{L} \to \mathfrak{M}$ be a mapping satisfying $g(0) = 0$*

$$\sup_{k\in\mathbb{N}} \|\mathcal{D}g(l_1, m_1), ..., \mathcal{D}g(l_k, m_k)\|_k \leq \alpha \left(\|l_1\|^a + \|m_1\|^a, .., \|l_k\|^a + \|m_k\|^a\right) \quad (3.6)$$

*for all $l_1, ..., l_k, m_1 ..., m_k \in \mathfrak{L}$. Then there exists a unique mapping $\mathfrak{T} : \mathfrak{L} \to \mathfrak{M}$ such that*

$$\sup_{k\in\mathbb{N}} \|(g(l_1) - \mathfrak{T}(l_1), ..., g(l_k) - \mathfrak{T}(l_k))\| \leq \frac{1}{2^{34} - 2^a}\Psi(\|l_1\|^a, ..., \|l_k\|^a) \quad (3.7)$$

*where*

$$\Psi = \frac{2}{34!}\alpha\left[\frac{1}{2}2^a + (17^a + 1) + 34(16^a + 1) + 561(15^a + 1) + 5984(14^a + 1) + 46376(13^a + 1)\right.$$
$$+278256(12^a + 1) + 1344904(11^a + 1) + 5379616(10^a + 1) + 18156204(9^a + 1)$$
$$+52451256(8^a + 1) + 131128140(7^a + 1) + 286097760(6^a + 1) + 548354040(5^a + 1)$$
$$\left.+927983760(4^a + 1) + 1391975640(3^a + 1) + 1855967520(2^a + 1) + 3370764540\right]$$

*Proof.* Proof is similar to that of Theorem 3.1 replacing $\eta$ by $\alpha\left(\|l_1\|^a + \|m_1\|^a, .., \|l_k\|^a + \|m_k\|^a\right)$ we arrive the result. □

**Corollary 3.3.** *Let $\mathfrak{L}$ be a linear space, and let $(\mathfrak{M}^k, \|.\|_k)$ be a multi-Banach space. Let $\alpha > 0, 0 < r + s = a < 34$ and $g : \mathfrak{L} \to \mathfrak{M}$ be a mapping satisfying $g(0) = 0$*

$$\sup_{k\in\mathbb{N}} \|\mathcal{D}g(l_1, m_1), ..., \mathcal{D}g(l_k, m_k)\|_k \leq \alpha \left(\|l_1\|^r . \|m_1\|^s, .., \|l_k\|^r . \|m_k\|^s\right) \quad (3.8)$$

*for all $l_1, ..., l_k, m_1 ..., m_k \in \mathfrak{L}$. Then there exists a unique mapping $\mathfrak{T} : \mathfrak{L} \to \mathfrak{M}$ such that*

$$\sup_{k\in\mathbb{N}} \|(g(l_1) - \mathfrak{T}(l_1), ..., g(l_k) - \mathfrak{T}(l_k))\| \leq \frac{1}{2^{34} - 2^a}\Psi_{34}(\|x_1\|^{r+s}, ..., \|x_k\|^{r+s}) \quad (3.9)$$

*where*

$$\Psi_{34} = \frac{2}{34!}\alpha\left[17^r + 34(16^r) + 561(15^r) + 5984(14^r) + 46376(13^r) + 278256(12^r) + 1344904(11^r)\right.$$
$$+5379616(10^r) + 18156204(9^r) + 52451256(8^r) + 131128140(7^r) + 286097760(6^r)$$
$$\left.+548354040(5^r) + 927983760(4^r) + 1391975640(3^r) + 1855967520(2^r) + 2203961430\right]$$

*Proof.* Proof is similar to that of Theorem 3.1 replacing $\eta$ by $\alpha\left(\|x_1\|^r . \|y_1\|^s, .., \|x_k\|^r . \|y_k\|^s\right)$ we arrive the result. □

**Example 3.4.** Let $k \in \mathbb{N}$. We define $\phi : \mathbb{R} \to \mathbb{R}$, by

$$\phi(l) = \begin{cases} 1 & l \in [1, \infty) \\ l^{34} & l \in (-\infty, \infty) \\ -1 & l \in (-\infty, -1]. \end{cases}$$

We consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(l) = \sum_{n=0}^{\infty} \frac{\phi(4^n l)}{4^{34n}}, \quad (l \in \mathbb{R}).$$

Then $f$ satisfies the following functional inequality:

$$\|\mathcal{D}g(l_1, m_1)), ..., \mathcal{D}g(l_k, m_k)\|_k \leq \frac{2^{34} + 34!}{4^{34} - 1} 4^{102} \left(|l_1|^{34} +, .., |l_k|^{34} + |m_1|^{34} +, .., |m_k|^{34}\right)$$

for all $l_1, ..., l_k, m_1, .., m_k \in \mathbb{R}$.

*Proof.* We have

$$|g(l)| \leq \frac{4^{34}}{4^{34} - 1}$$

for all $l \in \mathbb{R}$. Therefore, we see that $g$ is bounded. Let $l, m \in \mathbb{R}$. If $|l|^{34} + |m|^{34} = 0$ or $|l|^{34} + |m|^{34} \geq \frac{1}{4^{34}}$, then

$$|\mathcal{D}g(l, m)| \leq \frac{(2^{34} + 34!)\, 4^{34}}{4^{34} - 1} \leq \frac{(2^{34} + 34!)\, 4^{34}}{4^{34} - 1} 4^{34} \left(|l|^{34} + |m|^{34}\right).$$

Now, suppose that $0 < |l|^{34} + |m|^{34} \leq \frac{1}{4^{34}}$. Then there exists a non-negative integer $k$ such that

$$\frac{1}{4^{34(k+2)}} \leq |l|^{34} + |m|^{34} < \frac{1}{4^{34(k+1)}}.$$

Hence, $4^k l < \frac{1}{4}$ and $4^k m < \frac{1}{4}$,
and
$4^n(l+17m), 4^n(l+16m), 4^n(l+15m), 4^n(l+14m), 4^n(l+13m), 4^n(l+12m),$
$4^n(l+11m), 4^n(l+10m), 4^n(l+9m), 4^n(l+8m), 4^n(l+7m)4^n(l+6m),$
$4^n(l+5m), 4^n(l+4m), 4^n(l+3m), 4^n(l+2m)4^n(l+m)4^n(l), 4^n(l-m),$
$4^n(l-2m), 4^n(l-3m), 4^n(l-4m), 4^n(l-5m), 4^n(l-6m), 4^n(l-7m),$
$4^n(l-8m), 4^n(l-9m), 4^n(l-10m), 4^n(l-11m), 4^n(l-12m), 4^n(l-13m),$
$4^n(l-14m), 4^n(l-15m), 4^n(l-16m), 4^n(l-17m) \in (-1,1)$
for all $n = 0, 1, ..., k-1$. Thus we get

$$\begin{aligned}
\frac{|\mathcal{D}g(l,m|)}{|l|^{34}+|m|^{34}} &\leq \sum_{n=k}^{\infty} \frac{2^{34}+34!}{4^{34n}(|l|^{34}+|m|^{34})} \\
&\leq \sum_{n=0}^{\infty} \frac{2^{34}+34!}{4^{34n}4^{34(k+2)}(|l|^{34}+|m|^{34})} 4^{68} \\
&\leq \sum_{n=0}^{\infty} \frac{2^{34}+34!}{4^{34n}} 4^{68} = \frac{2^{34}+34!}{4^{34}-1} 4^{102},
\end{aligned}$$

or

$$|\mathcal{D}g(l,m)| \leq\leq \frac{2^{34}+34!}{4^{34}-1} 4^{102}(|l|^{34}+|m|^{34}).$$

□

## 4 Hyers-Ulam Stability of functional equation in Menger Probabilistic Normed Spaces

**Theorem 4.1.** *Let* $(\mathfrak{L}, \mathfrak{F}, \tau)$ *be a menger probabilistic normed space and* $(\mathfrak{M}, \mathfrak{G}, \tau)$ *be a Menger probabilistic banach space. If* $g : \mathfrak{L} \to \mathfrak{M}$ *satisfies*

$$\mathfrak{G}_{\mathcal{D}g(l,m)}(t+s) \geq \mathfrak{F}_l(t)\tau\mathfrak{F}_m(s) \tag{4.1}$$

*for all* $t, s > 0$*, and there exists positive number* $k$*,* $0 < k < \frac{1}{2^{34}}$ *such that*

$$\mathfrak{F}_l(2t) \geq \mathfrak{F}_{kl}(t) \tag{4.2}$$

*Then there exists a unique Quattuortrigintic mapping* $\mathfrak{T} : \mathfrak{L} \to \mathfrak{M}$ *such that*

$$\mathfrak{G}_{g(l)-\mathfrak{T}(l)}(t) \geq \mathfrak{F}_l\left(\frac{1-1.717986918\times 10^{10}k}{8589934592k}\right) \tag{4.3}$$

*Proof.* From Theorem 2.1 and using (2.18), inequality (4.1) is leading to

$$\mathfrak{G}_{g(2l)-2^{34}g(l)}(2t) \geq \mathfrak{F}_l(t) \tag{4.4}$$

for all $l \in \mathfrak{L}$ and $t > 0$. Replacing $l$ by $(\frac{l}{2})$ in (4.4), we have

$$\mathfrak{G}_{g(l)-2^{34}g(\frac{l}{2})}(2t) \geq \mathfrak{F}_{\frac{l}{2}}(t) \tag{4.5}$$

for all $l \in \mathfrak{L}$ and $t > 0$. By definition and replacing $t$ with $\frac{t}{2}$ in (4.5), we obtain

$$\mathfrak{G}_{g(l)-2^{34}g(\frac{l}{2})}(t) \geq \mathfrak{F}_l(t) \tag{4.6}$$

for all $l \in \mathfrak{L}$ and $t > 0$. Let $\mathfrak{P} := \{u : \mathfrak{L} \to \mathfrak{M} : u(0) = 0\}$ and the generalized metric d defined on $\mathfrak{P}$ by

$$d(u, v) = inf\left\{\lambda \in [0, \infty] : \mathfrak{G}_{u(l)-v(l)}(\lambda t) \geq \mathfrak{F}_l(t)\right\}. \tag{4.7}$$

Then it is easy to show that $(\mathfrak{P}, d)$ is generalized complete metric space [20]. We define an operator $\mathfrak{J} : \mathfrak{P} \to \mathfrak{P}$ by

$$\mathfrak{J}v(l) = 1.717986918 \times 10^{10} v\left(\frac{l}{2}\right)$$

for all $l \in \mathfrak{L}$. We assume that $\mathfrak{J}$ is a strictly contractive operator. Indeed given $u, v \in \mathfrak{P}$, let $\lambda \in (0, \infty)$ be an arbitrary constant with $d(u, v) < \lambda$. From the definition of d, it follows that

$$\mathfrak{G}_{u(l)-v(l)}(\lambda t) \geq \mathfrak{F}_l(t)$$

for all $l \in \mathfrak{L}$ and $t > 0$. Therefore,

$$\begin{aligned}
\mathfrak{G}_{\mathfrak{J}u(l)-\mathfrak{J}v(l)}(2^{34}k\lambda t) &= \mathfrak{G}_{2^{34}u(\frac{l}{2})-2^{34}v(\frac{l}{2})}(2^{34}k\lambda t) \\
&= \mathfrak{G}_{u(\frac{l}{2})-v(\frac{l}{2})}(k\lambda t) \\
&\geq \mathfrak{F}_{\frac{l}{2}}(kt) \\
&\geq \mathfrak{F}_l(t)
\end{aligned}$$

for all $l \in \mathfrak{L}$ and $t > 0$. That means $d(\mathfrak{J}u, \mathfrak{J}v) < 1.717986918 \times 10^{10}k\lambda$ for all $u, v \in \mathfrak{P}$. Hence, $\mathfrak{J}$ is a strictly contractive operator on $\mathfrak{P}$ with the Lipschitz constant $L = 1.717986918 \times 10^{10}k$. It follows from (4.6) that $d(g, \mathfrak{J}g) \leq 8589934592k$. According to Theorem 1.6, there exists a unique fixed point $\mathfrak{T}$ of $\mathfrak{J}$ in the set $K := \{v \in \mathfrak{P} : d(u,v) < \infty\}$ that is the mapping $\mathfrak{T} : \mathfrak{L} \to \mathfrak{M}$ with

$$\mathfrak{T}\left(\frac{l}{2}\right) = \frac{1}{1.717986918 \times 10^{10}} \mathfrak{T}(l)$$

for all $l \in \mathfrak{L}$. This implies that there exists $\lambda \in [0, \infty]$ satisfying

$$\mathfrak{G}_{g(l)-\mathfrak{T}(l)}(\lambda t) \geq \mathfrak{F}_l(t)$$

for all $l \in \mathfrak{L}$ and $t > 0$. Moreover, $d(\mathfrak{J}^n g, \mathfrak{T}) \to 0$, which implies

$$\mathfrak{T}(l) = \lim_{n\to\infty} 2^{34n} g(\frac{l}{2^n})$$

for all $l \in \mathfrak{L}$. Also,

$$d(g, \mathfrak{T}) \leq \frac{d(g, \mathfrak{T}g)}{1-L} \leq \frac{2^{33}k}{1-2^{34}k}$$

with $g \in K$. Hence,

$$\mathfrak{G}_{g(l)-\mathfrak{T}(l)}\left(\frac{2^{33}kt}{1-2^{34}k}\right) \geq \mathfrak{F}_l(t)$$

that means

$$\mathfrak{G}_{g(l)-\mathfrak{T}(l)}(t) \geq \mathfrak{F}_l\left(\frac{1-2^{34}k}{2^{33}k}\right)$$

for all $l \in \mathfrak{L}$ and $t > 0$. Since, $\mathfrak{F}_{\frac{l}{2^n}}(t) \geq \mathfrak{F}_{k^n l}(t)$ and $\mathfrak{F}_{\frac{m}{2^n}}(t) \geq \mathfrak{F}_{k^n m}(t)$ then

$$\mathfrak{F}_{\frac{l}{2^n}(t)} . \mathfrak{F}_{\frac{m}{2^n}(t)} \geq \mathfrak{F}_{k^n l}(t) \tau \mathfrak{F}_{k^n m}(t)$$

for all $l \in \mathfrak{L}$ and $t, s > 0$. this implies

$$\lim_{n\to\infty} \mathfrak{F}_{k^n l}(t) . \mathfrak{F}_{k^n m}(t) = 1$$

and

$$\mathfrak{G}_{\mathcal{D}\mathfrak{T}(x,y)}(t+s) = 1$$

for all $l, m \in \mathfrak{L}$. Hence $\mathfrak{T}$ is a Quatturotrigintic mapping and the proof is complete. □

## 5 Conclusion

In this work , we introduced the Quattuortrigintic functional equation and obtain its general solution in section 2. In section 3, We investigated the Hyers Ulam stability of Quattuortrigintic functional equation in Multi-Banach Spaces and in section 4, We examined the Hyers Ulam Stability of Quattuortrigintic functional equation in Menger Probabilistic Normed spaces.

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